1

Matrix Analysis

Exercises 1.3.3

▪ **1(a)** Yes, as the three vectors are linearly independent and span three-dimensional space.

1(b) No, since they are linearly dependent

$$\begin{bmatrix} 3 \\ 2 \\ 5 \end{bmatrix} - 2\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

1(c) No, do not span three-dimensional space. Note, they are also linearly dependent.

▪ **2** Transformation matrix is

$$\mathbf{A} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & \sqrt{2} \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Rotates the $(\mathbf{e}_1, \mathbf{e}_2)$ plane through $\pi/4$ radians about the $\mathbf{e}_3$ axis.

▪ **3** By checking axioms (a)–(h) on p. 10 it is readily shown that all cubics $ax^3 + bx^2 + cx + d$ form a vector space. Note that the space is four dimensional.

3(a) All cubics can be written in the form

$$ax^3 + bx^2 + cx + d$$

and $\{1,\, x,\, x^2,\, x^3\}$ are a linearly independent set spanning four-dimensional space. Thus, it is an appropriate basis.

3(b) No, does not span the required four-dimensional space. Thus a general cubic cannot be written as a linear combination of

$$(1-x),\ (1+x),\ (1-x^3),\ (1+x^3)$$

as no term in x^2 is present.

3(c) Yes as linearly independent set spanning the four-dimensional space

$$a(1-x)+b(1+x)+c(x^2-x^3)+d(x^2+x^3)$$

$$=(a+b)+(b-a)x+(c+a)x^2+(d-c)x^3$$

$$\equiv \alpha+\beta x+\gamma x^2+\delta x^3$$

3(d) Yes as a linear independent set spanning the four-dimensional space

$$a(x-x^2)+b(x+x^2)+c(1-x^3)+d(1+x^3)$$

$$=(a+b)+(b-a)x+(c+d)x^2+(d-c)x^3$$

$$\equiv \alpha+\beta x+\gamma x^2+\delta x^3$$

3(e) No not linearly independent set as

$$(4x^3+1)=(3x^2+4x^3)-(3x^2+2x)+(1+2x)$$

■ **4** $x+2x^3$, $2x-3x^5$, $x+x^3$ form a linearly independent set and form a basis for all polynomials of the form $\alpha+\beta x^3+\gamma x^5$. Thus, S is the space of all odd quadratic polynomials. It has dimension 3.

Exercises 1.4.3

■ **5(a)** Characteristic polynomial is $\lambda^3 - p_1\lambda^2 - p_2\lambda - p_3$ with
$p_1 =$ trace $\mathbf{A} = 12$

$$\mathbf{B}_1 = \mathbf{A} - 12\mathbf{I} = \begin{bmatrix} -9 & 2 & 1 \\ 4 & -7 & -1 \\ 2 & 3 & -8 \end{bmatrix}$$

$$\mathbf{A}_2 = \mathbf{A}\ \mathbf{B}_1 = \begin{bmatrix} -17 & -5 & -7 \\ -18 & -30 & 7 \\ 2 & -5 & -33 \end{bmatrix}$$

$p_2 = \frac{1}{2}$ trace $\mathbf{A}_2 = -40$

$$\mathbf{B}_2 = \mathbf{A}_2 + 40\mathbf{I} = \begin{bmatrix} 23 & -5 & -7 \\ -18 & 10 & 7 \\ 2 & -5 & 7 \end{bmatrix}$$

$$\mathbf{A}_3 = \mathbf{A}\ \mathbf{B}_2 = \begin{bmatrix} 35 & 0 & 0 \\ 0 & 35 & 0 \\ 0 & 0 & 35 \end{bmatrix}$$

$p_3 = \frac{1}{3}$ trace $\mathbf{A}_3 = 35$

Thus, characteristic polynomial is

$$\lambda^3 - 12\lambda^2 + 40\lambda - 35$$

Note that $\mathbf{B}_3 = \mathbf{A}_3 - 35\mathbf{I} = 0$ confirming check.

5(b) Characteristic polynomial is $\lambda^4 - p_1\lambda^3 - p_2\lambda^2 - p_3\lambda - p_4$ with
$p_1 =$ trace $\mathbf{A} = 4$

$$\mathbf{B}_1 = \mathbf{A} - 4\mathbf{I} = \begin{bmatrix} -2 & -1 & 1 & 2 \\ 0 & -3 & 1 & 0 \\ -1 & 1 & -3 & 1 \\ 1 & 1 & 1 & -4 \end{bmatrix}$$

$$\mathbf{A}_2 = \mathbf{A}\ \mathbf{B}_1 = \begin{bmatrix} -3 & 4 & 0 & -3 \\ -1 & -2 & -2 & 1 \\ 2 & 0 & -2 & -5 \\ -3 & -3 & -1 & 3 \end{bmatrix} \quad \Rightarrow p_2 = \frac{1}{2} \text{ trace } \mathbf{A}_2 = -2$$

$$\mathbf{B}_2 = \mathbf{A}_2 + 2\mathbf{I} = \begin{bmatrix} -1 & 4 & 0 & -3 \\ -1 & 0 & -2 & 1 \\ 2 & 0 & 0 & -5 \\ -3 & -3 & -1 & 5 \end{bmatrix}$$

$$\mathbf{A}_3 = \mathbf{A}\ \mathbf{B}_2 = \begin{bmatrix} -5 & 2 & 0 & -2 \\ 1 & 0 & -2 & -4 \\ -1 & -7 & -3 & 4 \\ 0 & 4 & -2 & -7 \end{bmatrix} \quad \Rightarrow p_3 = \frac{1}{3}\text{ trace } \mathbf{A}_3 = -5$$

$$\mathbf{B}_3 = \mathbf{A}_3 + 5\mathbf{I} = \begin{bmatrix} 0 & 0 & 0 & -2 \\ 1 & 5 & -2 & -4 \\ -1 & -8 & 2 & 4 \\ 0 & 4 & -2 & -2 \end{bmatrix}$$

$$\mathbf{A}_4 = \mathbf{A}\ \mathbf{B}_3 = \begin{bmatrix} -2 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & -2 \end{bmatrix} \quad \Rightarrow p_4 = \frac{1}{4}\text{trace } \mathbf{A}_4 = -2$$

Thus, characteristic polynomial is $\lambda^4 - 4\lambda^3 + 2\lambda^2 + 5\lambda + 2$

Note that $\mathbf{B}_4 = \mathbf{A}_4 + 2\mathbf{I} = 0$ as required by check.

■ **6(a)** Eigenvalues given by $\begin{vmatrix} 1-\lambda & 1 \\ 1 & 1-\lambda \end{vmatrix} = \lambda^2 - 2\lambda = \lambda(\lambda - 2) = 0$

so eigenvectors are $\lambda_1 = 2, \lambda_2 = 0$

Eigenvectors given by corresponding solutions of

$$(1 - \lambda_i)e_{i1} + e_{i2} = 0$$
$$e_{i1} + (1 - \lambda_i)e_{i2} = 0$$

Taking $i = 1, 2$ gives the eigenvectors as

$$\mathbf{e}_1 = [1\ 1]^T, \mathbf{e}_2 = [1\ \ -1]^T \qquad (1)$$

6(b) Eigenvalues given by $\begin{vmatrix} 1-\lambda & 2 \\ 3 & 2-\lambda \end{vmatrix} = \lambda^2 - 3\lambda - 4 = (\lambda + 1)(\lambda - 4) = 0$

so eigenvectors are $\lambda_1 = 4, \lambda_2 = -1$

Eigenvectors given by corresponding solutions of

$$(l - \lambda_i)e_{i1} + 2e_{i2} = 0$$
$$3e_{i1} + (2 - \lambda_i)e_{i2} = 0$$

Taking $i = 1, 2$ gives the eigenvectors as

$$\mathbf{e}_1 = [2\ 3]^T, \mathbf{e}_2 = [1\ \ -1]^T$$

6(c) Eigenvalues given by

$$\begin{vmatrix} 1-\lambda & 0 & -4 \\ 0 & 5-\lambda & 4 \\ -4 & 4 & 3-\lambda \end{vmatrix} = \lambda^3 + 9\lambda^2 + 9\lambda - 81 = (\lambda - 9)(\lambda - 3)(\lambda + 3) = 0$$

So the eigenvalues are $\lambda_1 = 9, \lambda_2 = 3, \lambda_3 = -3$.
The eigenvectors are given by the corresponding solutions of

$$(1 - \lambda_i)e_{i1} + 0e_{i2} - 4e_{i3} = 0$$
$$0e_{i1} + (5 - \lambda_i)e_{i2} + 4e_{i3} = 0$$
$$-4e_{i1} + 4e_{i2} + (3 - \lambda_i)e_{i3} = 0$$

Taking $i = 1, \lambda_i = 9$ solution is

$$\frac{e_{11}}{8} = -\frac{e_{12}}{16} = \frac{e_{13}}{-16} = \beta_1 \qquad \Rightarrow \mathbf{e}_1 = [-1\ 2\ 2]^T$$

Taking $i = 2, \lambda_i = 3$ solution is

$$\frac{e_{21}}{-16} = -\frac{e_{22}}{16} = \frac{e_{23}}{8} = \beta_2 \qquad \Rightarrow \mathbf{e}_2 = [2\ 2\ \ -1]^T$$

Taking $i = 3, \lambda_i = -3$ solution is

$$\frac{e_{31}}{32} = -\frac{e_{32}}{16} = \frac{e_{33}}{32} = \beta_3 \qquad \Rightarrow \mathbf{e}_3 = [2\ \ -1\ 2]^T$$

6(d) Eigenvalues given by

$$\begin{vmatrix} 1-\lambda & 1 & 2 \\ 0 & 2-\lambda & 2 \\ -1 & 1 & 3-\lambda \end{vmatrix} = 0$$

Adding column 1 to column 2 gives

$$\begin{vmatrix} 1-\lambda & 2-\lambda & 2 \\ 0 & 2-\lambda & 2 \\ -1 & 0 & 3-\lambda \end{vmatrix} = (2-\lambda) \begin{vmatrix} 1-\lambda & 1 & 2 \\ 0 & 1 & 2 \\ -1 & 0 & 3-\lambda \end{vmatrix}$$

$$\underset{R_1 - R_2}{=} (2-\lambda) \begin{vmatrix} 1-\lambda & 0 & 0 \\ 0 & 1 & 2 \\ -1 & 0 & 3-\lambda \end{vmatrix} = (2-\lambda)(1-\lambda)(3-\lambda)$$

so the eigenvalues are $\lambda_1 = 3, \lambda_2 = 2, \lambda_3 = 1$.
Eigenvectors are the corresponding solutions of $(\mathbf{A} - \lambda_i \mathbf{I})\mathbf{e}_i = 0$
When $\lambda = \lambda_1 = 3$ we have

$$\begin{bmatrix} -2 & 1 & 2 \\ 0 & -1 & 2 \\ -1 & 1 & 0 \end{bmatrix} \begin{bmatrix} e_{11} \\ e_{12} \\ e_{13} \end{bmatrix} = 0$$

leading to the solution

$$\frac{e_{11}}{-2} = -\frac{e_{12}}{2} = \frac{e_{13}}{-1} = \beta_1$$

so the eigenvector corresponding to $\lambda_2 = 3$ is $\mathbf{e}_1 = \beta_1[2\ 2\ 1]^T, \beta_1$ constant.
When $\lambda = \lambda_2 = 2$ we have

$$\begin{bmatrix} -1 & 1 & 2 \\ 0 & 0 & 2 \\ -1 & 1 & 1 \end{bmatrix} \begin{bmatrix} e_{21} \\ e_{22} \\ e_{23} \end{bmatrix} = 0$$

leading to the solution

$$\frac{e_{21}}{-2} = -\frac{e_{22}}{2} = \frac{e_{23}}{0} = \beta_3$$

so the eigenvector corresponding to $\lambda_2 = 2$ is $\mathbf{e}_2 = \beta_2[1\ 1\ 0]^T, \beta_2$ constant.
When $\lambda = \lambda_3 = 1$ we have

$$\begin{bmatrix} 0 & 1 & 2 \\ 0 & 1 & 2 \\ -1 & 1 & 2 \end{bmatrix} \begin{bmatrix} e_{31} \\ e_{32} \\ e_{33} \end{bmatrix} = 0$$

leading to the solution

$$\frac{e_{31}}{0} = -\frac{e_{32}}{2} = \frac{e_{33}}{1} = \beta_1$$

so the eigenvector corresponding to $\lambda_3 = 1$ is $\mathbf{e}_3 = \beta_3[0\ \ -2\ 1]^T, \beta_3$ constant.

6(e) Eigenvalues given by

$$\begin{vmatrix} 5-\lambda & 0 & 6 \\ 0 & 11-\lambda & 6 \\ 6 & 6 & -2-\lambda \end{vmatrix} = \lambda^3 - 14\lambda^2 - 23\lambda - 686 = (\lambda - 14)(\lambda - 7)(\lambda + 7) = 0$$

so eigenvalues are $\lambda_1 = 14, \lambda_2 = 7, \lambda_3 = -7$

Eigenvectors are given by the corresponding solutions of

$$(5-\lambda_i)e_{i1}+0e_{i2}+6e_{i3}=0$$
$$0e_{i1}+(11-\lambda_i)e_{i2}+6e_{i3}=0$$
$$6e_{i1}+6e_{i2}+(-2-\lambda_i)e_{i3}=0$$

When $i=1,\lambda_1=14$ solution is

$$\frac{e_{11}}{12}=\frac{-e_{12}}{-36}=\frac{e_{13}}{18}=\beta_1\ \Rightarrow \mathbf{e}_1=[2\ 6\ 3]^T$$

When $i=2,\lambda_2=7$ solution is

$$\frac{e_{21}}{-72}=\frac{-e_{22}}{-36}=\frac{e_{23}}{-24}=\beta_2\ \Rightarrow \mathbf{e}_2=[6\ -3\ 2]^T$$

When $i=3,\lambda_3=-7$ solution is

$$\frac{e_{31}}{54}=\frac{-e_{32}}{-36}=\frac{e_{33}}{-108}=\beta_3\ \Rightarrow \mathbf{e}_3=[3\ 2\ -6]^T$$

6(f) Eigenvalues given by

$$\begin{vmatrix}1-\lambda & -1 & 0\\ 1 & 2-\lambda & 1\\ -2 & 1 & -1-\lambda\end{vmatrix} \underset{=}{R_1 + R_2} \begin{vmatrix}-1-\lambda & 0 & -1-\lambda\\ 1 & 2-\lambda & 1\\ -2 & 1 & -1-\lambda\end{vmatrix}$$
$$=(1+\lambda)\begin{vmatrix}-1 & 0 & 0\\ 1 & 2-\lambda & 0\\ -2 & 1 & 1-\lambda\end{vmatrix}=0,\ \text{i.e. } (1+\lambda)(2-\lambda)(1-\lambda)=0$$

so eigenvalues are $\lambda_1=2,\lambda_2=1,\lambda_3=-1$
Eigenvectors are given by the corresponding solutions of

$$(1-\lambda_i)e_{i1}-e_{i2}+0e_{i3}=0$$
$$e_{i1}+(2-\lambda_i)e_{i2}+e_{i3}=0$$
$$-2e_{i1}+e_{i2}-(1+\lambda_i)e_{i3}=0$$

Taking $i=1,2,3$ gives the eigenvectors as

$$\mathbf{e}_1=[-1\ 1\ 1]^T,\mathbf{e}_2=[1\ 0\ -1]^T,\mathbf{e}_3=[1\ 2\ -7]^T$$

6(g) Eigenvalues given by

$$\begin{vmatrix} 4-\lambda & 1 & 1 \\ 2 & 5-\lambda & 4 \\ -1 & -1 & -\lambda \end{vmatrix} \underset{=}{R_1 + (R_2 + R_3)} \begin{vmatrix} 5-\lambda & 5-\lambda & 5-\lambda \\ 2 & 5-\lambda & 4 \\ -1 & -1 & -\lambda \end{vmatrix}$$

$$= (5-\lambda) \begin{vmatrix} 1 & 0 & 0 \\ 2 & 3-\lambda & 2 \\ -1 & 0 & 1-\lambda \end{vmatrix} = (5-\lambda)(3-\lambda)(1-\lambda) = 0$$

so eigenvalues are $\lambda_1 = 5, \lambda_2 = 3, \lambda_3 = 1$

Eigenvectors are given by the corresponding solutions of

$$(4-\lambda_i)e_{i1} + e_{i2} + e_{i3} = 0$$
$$2e_{i1} + (5-\lambda_i)e_{i2} + 4e_{i3} = 0$$
$$-e_{i1} - e_{i2} - \lambda_i e_{i3} = 0$$

Taking $i = 1, 2, 3$ and solving gives the eigenvectors as

$$\mathbf{e}_1 = [2\ 3\ \ -1]^T, \mathbf{e}_2 = [1\ \ -1\ 0]^T, \mathbf{e}_3 = [0\ \ -1\ 1]^T$$

6(h) Eigenvalues given by

$$\begin{vmatrix} 1-\lambda & -4 & -2 \\ 0 & 3-\lambda & 1 \\ 1 & 2 & 4-\lambda \end{vmatrix} \underset{=}{R_1 + 2R_2} \begin{vmatrix} 1-\lambda & 2-2\lambda & 0 \\ 0 & 3-\lambda & 1 \\ 1 & 2 & 4-\lambda \end{vmatrix}$$

$$= (1-\lambda) \begin{vmatrix} 1 & 0 & 0 \\ 0 & 3-\lambda & 1 \\ 1 & 0 & 4-\lambda \end{vmatrix} = (1-\lambda)(3-\lambda)(4-\lambda) = 0$$

so eigenvalues are $\lambda_1 = 4, \lambda_2 = 3, \lambda_3 = 1$

Eigenvectors are given by the corresponding solutions of

$$(1-\lambda_i)e_{i1} - 4e_{i2} - 2e_{i3} = 0$$
$$2e_{i1} + (3-\lambda_i)e_{i2} + e_{i3} = 0$$
$$e_{i1} + 2e_{i2} + (4-\lambda_i)e_{i3} = 0$$

Taking $i = 1, 2, 3$ and solving gives the eigenvectors as

$$\mathbf{e}_1 = [2 \ -1 \ -1]^T, \mathbf{e}_2 = [2 \ -1 \ 0]^T, \mathbf{e}_3 = [4 \ -1 \ -2]^T$$

Exercises 1.4.5

- **7(a)** Eigenvalues given by

$$\begin{vmatrix} 2-\lambda & 2 & 1 \\ 1 & 3-\lambda & 1 \\ 1 & 2 & 2-\lambda \end{vmatrix} \underset{=}{R_1 - R_2} \begin{vmatrix} 1-\lambda & -1+\lambda & 0 \\ 0 & 3-\lambda & 1 \\ 1 & 2 & 2-\lambda \end{vmatrix}$$

$$= (1-\lambda) \begin{vmatrix} 1 & 0 & 0 \\ 1 & 4-\lambda & 1 \\ 1 & 3 & 2-\lambda \end{vmatrix} = (1-\lambda)[\lambda^2 - 6\lambda + 5] = (1-\lambda)(\lambda-1)(\lambda-5) = 0$$

so eigenvalues are $\lambda_1 = 5, \lambda_2 = \lambda_3 = 1$
The eigenvectors are the corresponding solutions of

$$(2-\lambda_i)e_{i1} + 2e_{i2} + e_{i3} = 0$$
$$e_{i1} + (3-\lambda_i)e_{i2} + e_{i3} = 0$$
$$e_{i1} + 2e_{i2} + (2-\lambda_i)e_{i3} = 0$$

When $i = 1, \lambda_1 = 5$ and solution is

$$\frac{e_{11}}{4} = \frac{-e_{12}}{-4} = \frac{e_{13}}{4} = \beta_1 \ \Rightarrow \ \mathbf{e}_1 = [1 \ 1 \ 1]^T$$

When $\lambda_2 = \lambda_3 = 1$ solution is given by the single equation

$$e_{21} + 2e_{22} + e_{23} = 0$$

Following the procedure of Example 1.6 we can obtain two linearly independent solutions. A possible pair are

$$\mathbf{e}_2 = [0 \ 1 \ 2]^T \ , \ \mathbf{e}_3 = [1 \ 0 \ -1]^T$$

7(b) Eigenvalues given by

$$\begin{vmatrix} -\lambda & -2 & -2 \\ -1 & 1-\lambda & 2 \\ -1 & -1 & 2-\lambda \end{vmatrix} = -\lambda^3 + 3\lambda^2 - 4 = -(\lambda+1)(\lambda-2)^2 = 0$$

so eigenvalues are $\lambda_1 = \lambda_2 = 2, \lambda_3 = -1$
The eigenvectors are the corresponding solutions of

$$-\lambda_i e_{i1} - 2e_{i2} - 2e_{i3} = 0$$
$$-e_{i1} + (1-\lambda_i)e_{i2} + 2e_{i3} = 0$$
$$-e_{i1} - e_{i2} + (2-\lambda_i)e_{i3} = 0$$

When $i = 3, \lambda_3 = -1$ corresponding solution is

$$\frac{e_{31}}{8} = \frac{-e_{32}}{-1} = \frac{e_{33}}{3} = \beta_3 \ \Rightarrow\ \mathbf{e}_3 = [8\ 1\ 3]^T$$

When $\lambda_1 = \lambda_2 = 2$ solution is given by

$$-2e_{21} - 2e_{22} - 2e_{23} = 0 \qquad (1)$$
$$-e_{21} - e_{22} + 2e_{23} = 0 \qquad (2)$$
$$-e_{21} - e_{22} = 0 \qquad (3)$$

From (1) and (2) $e_{23} = 0$ and it follows from (3) that $e_{21} = -e_{22}$. We deduce that there is only one linearly independent eigenvector corresponding to the repeated eigenvalues $\lambda = 2$. A possible eigenvector is

$$\mathbf{e}_2 = [1\ \ -1\ 0]^T$$

7(c) Eigenvalues given by

$$\begin{vmatrix} 4-\lambda & 6 & 6 \\ 1 & 3-\lambda & 2 \\ -1 & -5 & -2-\lambda \end{vmatrix} \overset{R_1 - 3R_3}{=} \begin{vmatrix} 1-\lambda & -3+3\lambda & 0 \\ 1 & 3-\lambda & 2 \\ -1 & -5 & -2-\lambda \end{vmatrix}$$
$$= (1-\lambda)\begin{vmatrix} 1 & -3 & 0 \\ 1 & 3-\lambda & 2 \\ -1 & -5 & -2-\lambda \end{vmatrix} = (1-\lambda)\begin{vmatrix} 1 & 0 & 0 \\ 1 & 6-\lambda & 2 \\ 1 & -8 & -2-\lambda \end{vmatrix}$$
$$= (1-\lambda)(\lambda^2 + \lambda + 4) = (1-\lambda)(\lambda-2)^2 = 0$$

so eigenvalues are $\lambda_1 = \lambda_2 = 2,\ \lambda_3 = 1$.
The eigenvectors are the corresponding solutions of

$$(4-\lambda_i)e_{i1} + 6e_{i2} + 6e_{i3} = 0$$
$$e_{i1} + (3-\lambda_i)e_{i2} + 2e_{i3} = 0$$
$$-e_{i1} - 5e_{i2} - (2+\lambda_i)e_{i3} = 0$$

When $i = 3, \lambda_3 = 1$ corresponding solution is

$$\frac{e_{31}}{4} = \frac{-e_{32}}{-1} = \frac{e_{33}}{-3} = \beta_3 \ \Rightarrow\ \mathbf{e}_3 = [4\ 1\ \ -3]^T$$

When $\lambda_1 = \lambda_2 = 2$ solution is given by

$$2e_{21} + 6e_{22} + 6e_{23} = 0$$
$$e_{21} + e_{22} + 2e_{23} = 0$$
$$-e_{21} - 5e_{22} - 4e_{23} = 0$$

so that

$$\frac{e_{21}}{6} = \frac{-e_{22}}{-2} = \frac{e_{23}}{-4} = \beta_2$$

leading to only one linearly eigenvector corresponding to the eigenvector $\lambda = 2$. A possible eigenvector is

$$\mathbf{e}_2 = [3\ 1\ \ -2]^T$$

7(d) Eigenvalues given by

$$\begin{vmatrix} 7-\lambda & -2 & -4 \\ 3 & -\lambda & -2 \\ 6 & -2 & -3-\lambda \end{vmatrix} \underline{\underline{R_1-2R_2}} \begin{vmatrix} 1-\lambda & -2+2\lambda & 0 \\ 3 & -\lambda & -2 \\ 6 & -2 & -3-\lambda \end{vmatrix}$$
$$= (1-\lambda)\begin{vmatrix} 1 & -2 & 0 \\ 3 & -\lambda & -2 \\ 6 & -2 & -3-\lambda \end{vmatrix} = (1-\lambda)\begin{vmatrix} 1 & 0 & 0 \\ 3 & 6-\lambda & -2 \\ 6 & 10 & -3-\lambda \end{vmatrix}$$
$$= (1-\lambda)(\lambda-2)(\lambda-1) = 0$$

so eigenvalues are $\lambda_1 = 2,\ \lambda_2 = \lambda_3 = 1$.
The eigenvectors are the corresponding solutions of

$$(7-\lambda_i)e_{i1} - 2e_{i2} - 4e_{i3} = 0$$
$$3e_{i1} - \lambda_i e_{i2} - 2e_{i3} = 0$$
$$6e_{i1} - 2e_{i2} - (3+\lambda_i)e_{i3} = 0$$

When $i = 1, \lambda_2 = 2$ and solution is

$$\frac{e_{11}}{6} = \frac{-e_{12}}{-3} = \frac{e_{13}}{6} = \beta_1 \;\Rightarrow\; \mathbf{e}_1 = [2\ 1\ 2]^T$$

When $\lambda_2 = \lambda_3 = 1$ the solution is given by the single equation

$$3e_{21} - e_{22} - 2e_{23} = 0$$

Following the procedures of Example 1.6 we can obtain two linearly independent solutions. A possible pair are

$$\mathbf{e}_2 = [0\ 2\ \ -1]^T\ , \mathbf{e}_3 = [2\ 0\ 3]^T$$

■ **8**

$$(\mathbf{A} - \mathbf{I}) = \begin{bmatrix} -4 & -7 & -5 \\ 2 & 3 & 3 \\ 1 & 2 & 1 \end{bmatrix}$$

Performing a series of row and column operators this may be reduced to the form $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$ indicating that $(\mathbf{A} - \mathbf{I})$ is of rank 2. Thus, the nullity $q = 3 - 2 = 1$ confirming that there is only one linearly independent eigenvector associated with the eigenvalue $\lambda = 1$. The eigenvector is given by the solution of

$$\begin{aligned} -4e_{11} - 7e_{12} - 5e_{13} &= 0 \\ 2e_{11} + 3e_{12} + 3e_{13} &= 0 \\ e_{11} + 2e_{12} + e_{13} &= 0 \end{aligned}$$

giving

$$\frac{e_{11}}{-3} = \frac{-e_{12}}{-1} = \frac{e_{13}}{1} = \beta_1 \;\Rightarrow\; \mathbf{e}_1 = [-3\ 1\ 1]^T$$

■ **9**

$$(\mathbf{A} - \mathbf{I}) = \begin{bmatrix} 1 & 1 & -1 \\ -1 & -1 & 1 \\ -1 & -1 & 1 \end{bmatrix}$$

Performing a series of row and column operators this may be reduced to the form $\begin{bmatrix}1&0&0\\0&0&0\\0&0&0\end{bmatrix}$ indicating that $(\mathbf{A}-\mathbf{I})$ is of rank 1. Then, the nullity of $q=3-1=2$ confirming that there are two linearly independent eigenvectors associated with the eigenvalue $\lambda=1$. The eigenvectors are given by the single equation

$$e_{11}+e_{12}-e_{13}=0$$

and two possible linearly independent eigenvectors are

$$\mathbf{e}_1=[1\ 0\ 1]^T \text{ and } \mathbf{e}_2=[0\ 1\ 1]^T$$

Exercises 1.4.8

- **10** These are standard results.

- **11(a)** (i) Trace $\mathbf{A}=4+5+0=9=$ sum eigenvalues;

  (ii) $\det\mathbf{A}=15=5\times3\times1=$ product eigenvalues;

  (iii) $\mathbf{A^{-1}}=\dfrac{1}{15}\begin{bmatrix}4&-1&-1\\-4&1&-14\\3&3&18\end{bmatrix}$. Eigenvalues given by

$$\begin{vmatrix}4-15\lambda&-1&-1\\-4&1-15\lambda&-14\\3&3&18-15\lambda\end{vmatrix}\underset{=}{C_3-C_2}\begin{vmatrix}4-15\lambda&-1&0\\-4&1-15\lambda&-15+15\lambda\\3&3&15-15\lambda\end{vmatrix}$$

$$=(15-15\lambda)\begin{vmatrix}4-15\lambda&-1&0\\-4&1-15\lambda&-1\\3&3&1\end{vmatrix}=(15-15\lambda)(15\lambda-5)(15\lambda-3)=0$$

confirming eigenvalues as $1,\frac{1}{3},\frac{1}{5}$.

(iv) $A^T = \begin{bmatrix} 4 & 2 & -1 \\ 1 & 5 & -1 \\ 1 & 4 & 0 \end{bmatrix}$ having eigenvalues given by

$$\begin{vmatrix} 4-\lambda & 2 & -1 \\ 1 & 5-\lambda & -1 \\ 1 & 4 & -\lambda \end{vmatrix} = (\lambda - 5)(\lambda - 3)(\lambda - 1) = 0$$

that is, eigenvalue as for $\mathbf{A}$.

11(b) (i) $2\mathbf{A} = \begin{bmatrix} 8 & 2 & 2 \\ 4 & 10 & 8 \\ -2 & -2 & 0 \end{bmatrix}$ having eigenvalues given by

$$\begin{vmatrix} 8-\lambda & 2 & 2 \\ 4 & 10-\lambda & 8 \\ -2 & -2 & -\lambda \end{vmatrix} \underset{=}{C_1 - C_2} \begin{vmatrix} 6-\lambda & 2 & 2 \\ -6+\lambda & 10-\lambda & 8 \\ 0 & -2 & -\lambda \end{vmatrix}$$

$$= (6-\lambda) \begin{vmatrix} 1 & 2 & 2 \\ -1 & 10-\lambda & 8 \\ 0 & -2 & -\lambda \end{vmatrix} = (6-\lambda) \begin{vmatrix} 1 & 2 & 2 \\ 0 & 12-\lambda & 10 \\ 0 & -2 & -\lambda \end{vmatrix}$$

$$= (6-\lambda)(\lambda - 10)(\lambda - 2) = 0$$

Thus eigenvalues are 2 times those of $\mathbf{A}$; namely 6, 10 and 2.

(ii) $\mathbf{A} + 2\mathbf{I} = \begin{bmatrix} 6 & 1 & 1 \\ 2 & 7 & 4 \\ -1 & -1 & 2 \end{bmatrix}$ having eigenvalues given by

$$\begin{vmatrix} 6-\lambda & 1 & 1 \\ 2 & 7-\lambda & 4 \\ -1 & -1 & 2-\lambda \end{vmatrix} = -\lambda^3 + 15\lambda^2 - 71\lambda + 105 = -(\lambda - 7)(\lambda - 5)(\lambda - 3) = 0$$

confirming the eigenvalues as $5+2, 3+2, 1+2$.

Likewise for $\mathbf{A} - 2\mathbf{I}$

(iii) $A^2 = \begin{bmatrix} 17 & 8 & 8 \\ 14 & 23 & 22 \\ -6 & -6 & -5 \end{bmatrix}$ having eigenvalues given by

$$\begin{vmatrix} 17-\lambda & 8 & 8 \\ 14 & 23-\lambda & 22-\lambda \\ -6 & -6 & -5-\lambda \end{vmatrix} \underset{=\!=}{R_1 + (R_2) + R_3)} \begin{vmatrix} 25-\lambda & 25-\lambda & 25-\lambda \\ 14 & 23-\lambda & 22 \\ -6 & -6 & -5-\lambda \end{vmatrix}$$

$$= (25-\lambda) \begin{vmatrix} 1 & 0 & 0 \\ 14 & 9-\lambda & 8 \\ -6 & 0 & 1-\lambda \end{vmatrix} = (25-\lambda)(9-\lambda)(1-\lambda) = 0$$

that is, eigenvalues $\mathbf{A}^2$ are 25, 9, 1 which are those of $\mathbf{A}$ squared.

■ **12** Eigenvalues of $\mathbf{A}$ given by

$$\begin{vmatrix} -3-\lambda & -3 & -3 \\ -3 & 1-\lambda & -1 \\ -3 & -1 & 1-\lambda \end{vmatrix} \underset{=\!=}{R_3 + R_2} \begin{vmatrix} -3-\lambda & -3 & -3 \\ -3 & 1-\lambda & -1 \\ 0 & -2+\lambda & 2-\lambda \end{vmatrix}$$

$$= (\lambda-2) \begin{vmatrix} -3-\lambda & -3 & -3 \\ -3 & 1-\lambda & -1 \\ 0 & 1 & -1 \end{vmatrix} \underset{=\!=}{C_3 + C_2}(\lambda-2) \begin{vmatrix} -3-\lambda & -3 & -6 \\ -3 & (1-\lambda) & -\lambda \\ 0 & 1 & 0 \end{vmatrix}$$

$$= -(\lambda-2)(\lambda+6)(\lambda-3) = 0$$

so eigenvalues are $\lambda_1 = -6, \lambda_2 = 3, \lambda_3 = 2$
Eigenvectors are given by corresponding solutions of

$$(-3-\lambda_i)e_{i1} - 3e_{i2} - 3e_{i3} = 0$$
$$-3e_{i1} + (1-\lambda_i)e_{i2} - e_{i3} = 0$$
$$-3e_{i1} - e_{i2} + (1-\lambda_i)e_{i3} = 0$$

Taking $i = 1, 2, 3$ gives the eigenvectors as

$$\mathbf{e}_1 = [2\ 1\ 1]^T,\ \mathbf{e}_2 = [-1\ 1\ 1]^T,\ \mathbf{e}_3 = [0\ 1 - 1]^T$$

It is readily shown that

$$\mathbf{e}_1^T\mathbf{e}_2 = \mathbf{e}_1^T\mathbf{e}_3 = \mathbf{e}_2^T\mathbf{e}_3 = 0$$

so that the eigenvectors are mutually orthogonal.

■ **13** Let the eigenvector be $\mathbf{e} = [a\ b\ c]^T$ then since the three vectors are mutually orthogonal

$$a + b - 2c = 0$$

$$a + b - c = 0$$

giving $c = 0$ and $a = -b$ so an eigenvector corresponding to $\lambda = 2$ is $\mathbf{e} = [1\ {-1}\ 0]^T$.

Exercises 1.5.3

■ **14** Taking $x^{(0)} = [1\ 1\ 1]^T$ iterations may then be tabulated as follows:

Iteration k	0	1	2	3	4
	1	0.9	0.874	0.869	0.868
$\mathbf{x}^{(k)}$	1	1	1	1	1
	1	0.5	0.494	0.493	0.492
	9	7.6	7.484	7.461	7.457
$\mathbf{A}\ \mathbf{x}^{(k)}$	10	8.7	8.61	8.592	8.589
	5	4.3	4.242	4.231	4.228
$\lambda \simeq$	10	8.7	8.61	8.592	8.589

Thus, estimate of dominant eigenvalue is $\lambda \simeq 8.59$ and corresponding eigenvector $\mathbf{x} \simeq [0.869\ 1\ 0.493]^T$or $\mathbf{x} \simeq [0.61\ 0.71\ 0.35]^T$ in normalised form.

■ **15(a)** Taking $\mathbf{x}^{(0)} = [1\ 1\ 1]^T$ iterations may then be tabulated as follows:

Iteration k	0	1	2	3	4	5	**6**
	1	0.75	0.667	0.636	0.625	0.620	0.619
$\mathbf{x}^{(k)}$	1	1	1	1	1	1	1
	1	1	1	1	1	1	1
	3	2.5	2.334	2.272	2.250	2.240	
$\mathbf{A}\ \mathbf{x}^{(k)}$	4	3.75	3.667	3.636	3.625	3.620	
	4	3.75	3.667	3.636	3.625	3.620	
$\lambda \simeq$	4	3.75	3.667	3.636	3.625	3.620	

Thus, correct to two decimal places dominant eigenvalue is 3.62 having corresponding eigenvectors $[0.62\ 1\ 1]^T$.

15(b) Taking $\mathbf{x}^{(0)} = [1\ 1\ 1]^T$ iterations may be tabulated as follows:

Iteration k	0	1	2	3	4	5
$\mathbf{x}^{(k)}$	1	0.364	0.277	0.257	0.252	0.251
	1	0.545	0.506	0.501	0.493	0.499
	1	1	1	1	1	1
$\mathbf{A}\ \mathbf{x}^{(k)}$	4	2.092	1.831	1.771	1.756	
	6	3.818	3.566	3.561	3.49	
	11	7.546	7.12	7.03	6.994	
$\lambda\ \simeq$	11	7.546	7.12	7.03	6.994	

Thus, correct to two decimal places dominant eigenvalue is 7 having corresponding eigenvector $[0.25\ 0.5\ 1]^T$.

15(c) Taking $\mathbf{x}^{(0)} = [1\ 1\ 1\ 1]^T$ iterations may then be tabulated as follows:

Iteration k	0	1	2	3	4	5	6
$\mathbf{x}^{(k)}$	1	1	1	1	1	1	1
	1	0	−0.5	−0.6	−0.615	−0.618	− 0.618
	1	1	−0.5	−0.6	−0.615	−0.618	−0.618
	1	1	1	1	1	1	1
$\mathbf{A}\ \mathbf{x}^{(k)}$	1	2	2.5	2.6	2.615	2.618	
	0	−1	−1.5	−1.6	−1.615	−1.618	
	0	−1	−1.5	−1.6	−1.615	−1.618	
	1	2	2.5	2.6	2.615	2.618	
$\lambda\ \simeq$	1	2	2.5	2.6	2.615	2.618	

Thus, correct to two decimal places dominant eigenvalue is 2.62 having corresponding eigenvector $[1\ \ -0.62\ \ -0.62\ \ \ 1]^T$.

- **16** The eigenvalue λ_1 corresponding to the dominant eigenvector $\mathbf{e}_1 = [1\ 1\ 2]^T$ is such that $\mathbf{A}\ \mathbf{e}_1 = \lambda_1 \mathbf{e}_1$ so

$$\begin{bmatrix} 3 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} = \lambda_1 \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$$

so $\lambda_1 = 6$.

Then

$$\mathbf{A}_1 = \mathbf{A} - 6\hat{\mathbf{e}}_1\hat{\mathbf{e}}_1^T \text{ where } \hat{\mathbf{e}}_1 = [\frac{1}{\sqrt{6}}\frac{1}{\sqrt{6}}\frac{2}{\sqrt{6}}]^T$$

so

$$\mathbf{A}_1 = \begin{bmatrix} 3 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 5 \end{bmatrix} - \begin{bmatrix} 1 & 1 & 2 \\ 1 & 1 & 2 \\ 2 & 2 & 4 \end{bmatrix} = \begin{bmatrix} 2 & 0 & -1 \\ 0 & 2 & -1 \\ -1 & -1 & 1 \end{bmatrix}$$

Applying the power method with $\mathbf{x}^{(0)} = [1\ 1\ 1]^T$

$$\mathbf{y}^{(1)} = \mathbf{A}_1\mathbf{x}^{(0)} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} = \mathbf{x}^{(1)}$$

$$\mathbf{y}^{(2)} = \mathbf{A}_1\mathbf{x}^{(1)} = \begin{bmatrix} 3 \\ 3 \\ -3 \end{bmatrix} = 3\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$$

Clearly, $\lambda_2 = 3$ and $\hat{\mathbf{e}}_2 = \frac{1}{\sqrt{3}}[1\ 1\ -1]^T$.
Repeating the process

$$\mathbf{A}_2 = \mathbf{A}_1 - \lambda_2\hat{\mathbf{e}}_2\hat{\mathbf{e}}_2^T = \begin{bmatrix} 2 & 0 & -1 \\ 0 & 2 & -1 \\ -1 & -1 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 1 & -1 \\ 1 & 1 & -1 \\ -1 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Taking $\mathbf{x}^{(0)} = [1\ -1\ 0]^T$ the power method applied to $\mathbf{A}_2$ gives

$$\mathbf{y}^{(1)} = \mathbf{A}_2\mathbf{x}^{(0)} = \begin{bmatrix} 2 \\ -2 \\ 0 \end{bmatrix} = 2\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$$

and clearly, $\lambda_3 = 2$ with $\hat{\mathbf{e}}_3 = \frac{1}{\sqrt{2}}[1\ -1\ 0]^T$.

■ **17** The three Gerschgorin circles are

$$|\ \lambda - 5\ | = 2,\ \ |\ \lambda\ | = 2,\ \ |\ \lambda + 5\ | = 2$$

which are three non-intersecting circles. Since the given matrix **A** is symmetric its three eigenvalues are real and it follows from Theorem 1.2 that

$$3 < \lambda_1 < 7\ ,\ \ -2 < \lambda_2 < 2\ ,\ \ -7 < \lambda_3 < 7$$

(i.e., an eigenvalue lies within each of the three circles).

■ **18** The characteristic equation of the matrix **A** is

$$\begin{vmatrix} 10-\lambda & -1 & 0 \\ -1 & 2-\lambda & 2 \\ 0 & 2 & 3-\lambda \end{vmatrix} = 0$$

$$\text{that is}\quad (10-\lambda)[(2-\lambda)(3-\lambda)-4]-(3-\lambda)=0$$
$$\text{or}\quad f(\lambda)=\lambda^3-15\lambda^2+51\lambda-17=0$$

Taking $\lambda_0 = 10$ as the starting value the Newton–Raphson iterative process produces the following table:

i	λ_i	$f(\lambda_i)$	$f'(\lambda_i)$	$-\dfrac{f(\lambda_i)}{f'(\lambda_i)}$
0	10	7	−51.00	0.13725
1	10.13725	−0.28490	−55.1740	−0.00516
2	10.13209	−0.00041	−55.0149	−0.000007

Thus to three decimal places the largest eigenvalue is $\lambda = 10.132$

Using Properties 1.1 and 1.2 of section 1.4.6 we have

$$\sum_{i=1}^{3}\lambda_i = \text{ trace }\mathbf{A} = 15 \text{ and } \prod_{i=1}^{3}\lambda_i = |\ \mathbf{A}\ | = 17$$

Thus,

$$\lambda_2 + \lambda_3 = 15 - 10.132 = 4.868$$
$$\lambda_2\lambda_3 = 1.67785$$
$$\text{so}\quad \lambda_2(4.868 - \lambda_2) = 1.67785$$
$$\lambda_2^2 - 4.868\lambda_2 + 1.67785 = 0$$
$$\lambda_2 = 2.434 \pm 2.0607$$
$$\text{that is}\quad \lambda_2 = 4.491 \text{ and } \quad \lambda_3 = 0.373$$

- **19(a)** If $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$ are the corresponding eigenvectors to $\lambda_1, \lambda_2, \ldots, \lambda_n$ then $(K\mathbf{I} - \mathbf{A})\mathbf{e}_i = (K - \lambda_i)\mathbf{e}_i$ so that $\mathbf{A}$ and $(K\mathbf{I} - \mathbf{A})$ have the same eigenvectors and eigenvalues differ by K.

Taking $\mathbf{x}^{(o)} = \sum_{i=1}^{n} \alpha_r \mathbf{e}_i$ then

$$\mathbf{x}^{(p)} = (K\mathbf{I} - \mathbf{A})\mathbf{x}^{(p-1)} = (K\mathbf{I} - \mathbf{A})^2\mathbf{x}^{(p-2)} = \ldots = \sum_{r=1}^{n} \alpha_r (K - \lambda_r)^p \mathbf{e}_r$$

Now $K - \lambda_n > K - \lambda_{n-1} > \ldots > K - \lambda_1$ and

$$\begin{aligned}\mathbf{x}^{(p)} &= \alpha_n (K - \lambda_n)^p \mathbf{e}_n + \sum_{r=1}^{n} \alpha_r (K - \lambda_r)^p \mathbf{e}_r \\ &= (K - \lambda_n)^p [\alpha_n \mathbf{e}_n + \sum_{r=1}^{n-1} \alpha_r \left[\frac{K - \lambda_r}{K - \lambda_n}\right]^p \mathbf{e}_r] \\ &\to (K - \lambda_n)^p \alpha_n \mathbf{e}_n = K\mathbf{e}_n \text{ as } p \to \infty\end{aligned}$$

Also

$$\frac{x_i^{(p+1)}}{x_i^{(p)}} \to \frac{(K - \lambda_n)^{p+1}}{(K - \lambda_n)^p} \frac{\alpha_n e_n}{\alpha_n e_n} = K - \lambda_n$$

Hence, we can find λ_n

19(b) Since $\mathbf{A}$ is a symmetric matrix its eigenvalues are real. By Gerschgorin's theorem the eigenvalues lie in the union of the intervals

$$|\, \lambda - 2 \,| \leq 1, \;\; |\, \lambda - 2 \,| \leq 2, \;\; |\, \lambda - 2 \,| \leq 1$$
$$\text{i.e.}\quad |\, \lambda - 2 \,| \leq 2 \text{ or } 0 \leq \lambda \leq 4.$$

Taking $K = 4$ in (a)

$$K\mathbf{I} - \mathbf{A} = 4\mathbf{I} - \mathbf{A} = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix}$$

Taking $\mathbf{x}^{(0)} = [1\ 1\ 1]^T$ iterations using the power method are tabulated as follows:

Iteration k	0	1	2	3	4	5
	1	0.75	0.714	0.708	0.707	0.707
$\mathbf{x}^{(k)}$	1	1	1	1	1	1
	1	0.75	0.714	0.708	0.707	0.707
	3	2.5	2.428	2.416	2.414	
$\mathbf{A}\ \mathbf{x}^{(k)}$	4	3.5	3.428	3.416	3.414	
	3	2.5	2.428	2.416	2.414	
$\lambda \simeq$	4	3.5	3.428	3.416	3.414	

Thus $\lambda_3 = 4 - 3.41 = 0.59$ correct to two decimal places.

Exercises 1.6.3

■ **20** Eigenvalues given by

$$\Delta = \begin{vmatrix} -1-\lambda & 6 & -12 \\ 0 & -13-\lambda & 30 \\ 0 & -9 & 20-\lambda \end{vmatrix} = 0$$

Now $\Delta = (-1-\lambda)\begin{vmatrix} -13-\lambda & 30 \\ -9 & 20-\lambda \end{vmatrix} = (-1-\lambda)(\lambda^2 - 7\lambda + 10)$
$= (-1-\lambda)(\lambda-5)(\lambda-2)$ so $\Delta = 0$ gives $\lambda_1 = 5, \lambda_2 = 2, \lambda_3 = -1$
Corresponding eigenvectors are given by the solutions of

$$(\mathbf{A} - \lambda_i\mathbf{I})\mathbf{e}_i = 0$$

When $\lambda = \lambda_1 = 5$ we have

$$\begin{bmatrix} -6 & 6 & -12 \\ 0 & -18 & 30 \\ 0 & -9 & 15 \end{bmatrix} \begin{bmatrix} e_{11} \\ e_{12} \\ e_{13} \end{bmatrix} = 0$$

leading to the solution

$$\frac{e_{11}}{-36} = \frac{-e_{12}}{-180} = \frac{e_{13}}{108} = \beta_1$$

so the eigenvector corresponding to $\lambda_1 = 5$ is $\mathbf{e}_1 = \beta_1[1 \ -5 \ -3]^T$

When $\lambda = \lambda_2 = 2$, we have

$$\begin{bmatrix} -3 & 6 & -12 \\ 0 & -15 & 30 \\ 0 & -9 & 18 \end{bmatrix} \begin{bmatrix} e_{21} \\ e_{22} \\ e_{23} \end{bmatrix} = 0$$

leading to the solution

$$\frac{e_{21}}{0} = \frac{-e_{22}}{-90} = \frac{e_{23}}{45} = \beta_2$$

so the eigenvector corresponding to $\lambda_2 = 2$ is $\mathbf{e}_2 = \beta_2[0 \ 2 \ 1]^T$

When $\lambda = \lambda_3 = -1$, we have

$$\begin{bmatrix} 0 & 6 & -12 \\ 0 & -12 & 30 \\ 0 & -9 & 21 \end{bmatrix} \begin{bmatrix} e_{31} \\ e_{32} \\ e_{33} \end{bmatrix} = 0$$

leading to the solution

$$\frac{e_{31}}{18} = \frac{-e_{32}}{0} = \frac{e_{33}}{0} = \beta_3$$

so the eigenvector corresponding to $\lambda_3 = -1$ is $\mathbf{e}_3 = \beta_3[1 \ 0 \ 0]^T$

A modal matrix $\mathbf{M}$ and spectral matrix $\mathbf{\Lambda}$ are

$$\mathbf{M} = \begin{bmatrix} 1 & 0 & 1 \\ -5 & 2 & 0 \\ -3 & 1 & 0 \end{bmatrix} \qquad \mathbf{\Lambda} = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

$$\mathbf{M}^{-1} = \begin{bmatrix} 0 & 1 & -2 \\ 0 & 3 & -5 \\ 1 & -1 & 2 \end{bmatrix}$$ and matrix multiplication confirms $\mathbf{M}^{-1}\mathbf{A}\,\mathbf{M} = \mathbf{\Lambda}$

■ **21** From Example 1.9 the eigenvalues and corresponding normalised eigenvectors of $\mathbf{A}$ are

$$\lambda_1 = 6, \lambda_2 = 3, \lambda_3 = 1$$

$$\hat{\mathbf{e}}_1 = \frac{1}{\sqrt{5}}[1 \ 2 \ 0]^T, \hat{\mathbf{e}}_2 = [0 \ 0 \ 1]^T, \hat{\mathbf{e}}_3 = \frac{1}{\sqrt{5}}[-2 \ 1 \ 0]^T,$$

$$\hat{\mathbf{M}} = \frac{1}{\sqrt{5}} \begin{bmatrix} 1 & 0 & -2 \\ 2 & 0 & 1 \\ 0 & \sqrt{5} & 0 \end{bmatrix}$$

$$\hat{\mathbf{M}}^T\ \mathbf{A}\ \mathbf{M} = \frac{1}{5}\begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & \sqrt{5} \\ -2 & 1 & 0 \end{bmatrix}\begin{bmatrix} 2 & 2 & 0 \\ 2 & 5 & 0 \\ 0 & 0 & 3 \end{bmatrix}\begin{bmatrix} 1 & 0 & -2 \\ 2 & 0 & 1 \\ 0 & \sqrt{5} & 0 \end{bmatrix}$$

$$= \frac{1}{5}\begin{bmatrix} 6 & 12 & 0 \\ 0 & 0 & 3\sqrt{5} \\ -2 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 & -2 \\ 2 & 0 & 1 \\ 0 & \sqrt{5} & 0 \end{bmatrix}$$

$$= \frac{1}{5}\begin{bmatrix} 30 & 0 & 0 \\ 0 & 15 & 0 \\ 0 & 0 & 5 \end{bmatrix} = \begin{bmatrix} 6 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \mathbf{\Lambda}$$

■ **22** The eigenvalues of $\mathbf{A}$ are given by

$$\begin{vmatrix} 5-\lambda & 10 & 8 \\ 10 & 2-\lambda & -2 \\ 8 & -2 & 11-\lambda \end{vmatrix} = -(\lambda^3 - 18\lambda^2 - 81\lambda + 1458) = -(\lambda-9)(\lambda+9)(\lambda-18) = 0$$

so eigenvalues are $\lambda_1 = 18, \lambda_2 = 9, \lambda_3 = -9$

The eigenvectors are given by the corresponding solutions of

$$(5-\lambda_i)e_{i1} + 10e_{i2} + 8e_{i3} = 0$$
$$10e_{i1} + (2-\lambda_i)e_{i2} - 2e_{i3} = 0$$
$$8e_{i1} - 2e_{i2} + (11-\lambda_i)e_{i3} = 0$$

Taking $i = 1, 2, 3$ and solving gives the eigenvectors as

$$\mathbf{e}_1 = [2\ 1\ 2]^T,\ \mathbf{e}_2 = [1\ 2\ -2]^T, \mathbf{e}_3 = [-2\ 2\ 1]^T$$

Corresponding normalised eigenvectors are

$$\hat{\mathbf{e}}_1 = \frac{1}{3}[2\ 1\ 2]^T,\ \hat{\mathbf{e}}_2 = \frac{1}{3}[1\ 2\ -2]^T,\ \hat{\mathbf{e}}_3 = \frac{1}{3}[-2\ 2\ 1]^T$$

$$\hat{\mathbf{M}} = \frac{1}{3}\begin{bmatrix} 2 & 1 & -2 \\ 1 & 2 & 2 \\ 2 & -2 & 1 \end{bmatrix},\quad \hat{\mathbf{M}}^T = \frac{1}{3}\begin{bmatrix} 2 & 1 & 2 \\ 1 & 2 & -2 \\ -2 & 2 & 1 \end{bmatrix}$$

$$\hat{\mathbf{M}}^T\ \mathbf{A}\ \mathbf{M} = \frac{1}{9}\begin{bmatrix} 2 & 1 & 2 \\ 1 & 2 & -2 \\ -2 & 2 & 1 \end{bmatrix}\begin{bmatrix} 5 & 10 & 8 \\ 10 & 2 & -2 \\ 8 & -2 & 11 \end{bmatrix}\begin{bmatrix} 2 & 1 & -2 \\ 1 & 2 & 2 \\ 2 & -2 & 1 \end{bmatrix}$$

$$= \frac{1}{9}\begin{bmatrix} 36 & 18 & 36 \\ 9 & 18 & -18 \\ 18 & -18 & -9 \end{bmatrix}\begin{bmatrix} 2 & 1 & -2 \\ 1 & 2 & 2 \\ 2 & -2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 2 & 4 \\ 1 & 2 & -2 \\ 2 & -2 & -1 \end{bmatrix}\begin{bmatrix} 2 & 1 & -2 \\ 1 & 2 & 2 \\ 2 & -2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 18 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & -9 \end{bmatrix} = \mathbf{\Lambda}$$

23

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & -2 \\ -1 & 2 & 1 \\ 0 & 1 & -1 \end{bmatrix}$$

Eigenvalues given by

$$0 = \begin{vmatrix} 1-\lambda & 1 & -2 \\ -1 & 2-\lambda & 1 \\ 0 & 1 & -1-\lambda \end{vmatrix} = -(\lambda^3 - 2\lambda^2 - \lambda + 2) = -(\lambda-1)(\lambda-2)(\lambda+1) = 0$$

so eigenvalues are $\lambda_1 = 2, \lambda_2 = 1, \lambda_3 = -1$.
The eigenvectors are given by the corresponding solutions of

$$(1-\lambda_i)e_{i1} + e_{i2} - 2e_{i3} = 0$$

$$-e_{i1} + (2-\lambda_i)e_{i2} + e_{i3} = 0$$

$$0e_{i1} + e_{i2} - (1+\lambda_i)e_{i3} = 0$$

Taking $i = 1, 2, 3$ and solving gives the eigenvectors as

$$\mathbf{e}_1 = [1\ 3\ 1]^T, \mathbf{e}_2 = [3\ 2\ 1]^T, \mathbf{e}_3 = [1\ 0\ 1]^T$$

$$\mathbf{M} = \begin{bmatrix} 1 & 3 & 1 \\ 3 & 2 & 0 \\ 1 & 1 & 1 \end{bmatrix}, \quad \mathbf{\Lambda} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

$$\mathbf{M}^{-1} = -\frac{1}{6}\begin{bmatrix} 2 & -2 & -2 \\ -3 & 0 & -3 \\ 1 & 2 & -7 \end{bmatrix}$$

Matrix multiplication then confirms

$$\mathbf{M}^{-1}\ \mathbf{A}\ \mathbf{M} = \mathbf{\Lambda} \quad \text{and} \quad \mathbf{A} = \mathbf{M}\ \mathbf{\Lambda}\ \mathbf{M}^{-1}$$

■ **24** Eigenvalues given by

$$\begin{vmatrix} 3-\lambda & -2 & 4 \\ -2 & -2-\lambda & 6 \\ 4 & 6 & -1-\lambda \end{vmatrix} = -\lambda^3 + 63\lambda - 162 = -(\lambda+9)(\lambda-6)(\lambda-3) = 0$$

so the eigenvalues are $\lambda_1 = -9, \lambda_2 = 6, \lambda_3 = 3$. The eigenvectors are the corresponding solutions of

$$(3-\lambda_i)e_{i1} - 2e_{i2} + 44e_{i3} = 0$$
$$-2e_{i1} - (2+\lambda_i)e_{i2} + 6e_{i3} = 0$$
$$4e_{i1} + 6e_{i2} - (1+\lambda_i)e_{i3} = 0$$

Taking $i = 1, 2, 3$ and solving gives the eigenvectors as

$$\mathbf{e}_1 = [1\ 2\ \ -2]^T, \mathbf{e}_2 = [2\ 1\ 2]^T, \mathbf{e}_3 = [-2\ 2\ 1]^T$$

Since $\mathbf{e}_1^T\mathbf{e}_2 = \mathbf{e}_1^T\mathbf{e}_3 = \mathbf{e}_2^T\mathbf{e}_3 = 0$ the eigenvectors are orthogonal

$$\mathbf{L} = [\hat{\mathbf{e}}_1\ \hat{\mathbf{e}}_2\ \hat{\mathbf{e}}_3] = \frac{1}{3}\begin{bmatrix} 1 & 2 & -2 \\ 2 & 1 & 2 \\ -2 & 2 & 1 \end{bmatrix}$$

$$\begin{aligned}\hat{\mathbf{L}}\ \mathbf{A}\ \mathbf{L} &= \frac{1}{9}\begin{bmatrix} 1 & 2 & -2 \\ 2 & 1 & 2 \\ -2 & 2 & 1 \end{bmatrix}\begin{bmatrix} 3 & -2 & 4 \\ -2 & -2 & 6 \\ 4 & 6 & -1 \end{bmatrix}\begin{bmatrix} 1 & 2 & -2 \\ 2 & 1 & 2 \\ -2 & 2 & 1 \end{bmatrix} \\ &= \frac{1}{9}\begin{bmatrix} -9 & -18 & 18 \\ 12 & 6 & 12 \\ -6 & 6 & 3 \end{bmatrix}\begin{bmatrix} 1 & 2 & -2 \\ 2 & 1 & 2 \\ -2 & 2 & 1 \end{bmatrix} \\ &= \frac{1}{9}\begin{bmatrix} -81 & 0 & 0 \\ 0 & 54 & 0 \\ 0 & 0 & 27 \end{bmatrix} = \begin{bmatrix} -9 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 3 \end{bmatrix} = \mathbf{\Lambda}\end{aligned}$$

■ **25** Since the matrix $\mathbf{A}$ is symmetric the eigenvectors

$$\mathbf{e}_1 = [1\ 2\ 0]^T,\ \mathbf{e}_2 = [-2\ 1\ 0]^T,\ \mathbf{e}_3 = [e_{31}\ e_{32}\ e_{33}]^T$$

are orthogonal. Hence,

$$\mathbf{e}_1^T\mathbf{e}_3 = e_{31} + 2e_{32} = 0 \text{ and } \mathbf{e}_2^T\mathbf{e}_3 = -2e_{31} + e_{32} = 0$$

Thus, $e_{31} = e_{32} = 0$ and e_{33} arbitrary so a possible eigenvector is $\mathbf{e}_3 = [0\ 0\ 1]^T$.
Using $\mathbf{A} = \hat{\mathbf{M}}\ \mathbf{\Lambda}\ \hat{\mathbf{M}}^T$ where $\mathbf{\Lambda} = \begin{bmatrix} 6 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ gives

$$\begin{aligned}\mathbf{A} &= \begin{bmatrix} \frac{1}{\sqrt{5}} & -\frac{2}{\sqrt{5}} & 0 \\ \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 6 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} & 0 \\ -\frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 2 & 2 & 0 \\ 2 & 5 & 0 \\ 0 & 0 & 3 \end{bmatrix}\end{aligned}$$

■ **26** $\mathbf{A} - \mathbf{I} = \begin{bmatrix} -4 & -7 & -5 \\ 2 & 3 & 3 \\ 1 & 2 & 1 \end{bmatrix} \sim \begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ is of rank 2

Nullity $(\mathbf{A} - \mathbf{I}) = 3 - 2 = 1$ so there is only one linearly independent vector corresponding to the eigenvalue 1. The corresponding eigenvector $\mathbf{e}_1$ is given by the solution of $(\mathbf{A} - \mathbf{I})\mathbf{e}_1 = 0$ or

$$\begin{aligned} -4e_{11} - 7e_{12} - 5e_{13} &= 0 \\ 2e_{11} + 3e_{12} + 3e_{13} &= 0 \\ e_{11} + 2e_{12} + 2_{12} &= 0 \end{aligned}$$

that is, $\mathbf{e}_1 = [-3\ 1\ 1]^T$. To obtain the generalised eigenvector $\mathbf{e}_1^*$ we solve

$$(\mathbf{A} - \mathbf{I})\mathbf{e}_1^* = \mathbf{e}_1 \text{ or}$$
$$\begin{bmatrix} -4 & -7 & -5 \\ 2 & 3 & 3 \\ 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} e_{11}^* \\ e_{12}^* \\ e_{13}^* \end{bmatrix} = \begin{bmatrix} -3 \\ 1 \\ 1 \end{bmatrix}$$

giving $\mathbf{e}_1^* = [-1\ 1\ 0]^T$. To obtain the second generalised eigenvector $\mathbf{e}_1^{**}$ we solve

$$(\mathbf{A} - \mathbf{I})\mathbf{e}_1^{**} = \mathbf{e}_1^* \text{ or}$$
$$\begin{bmatrix} -4 & -7 & -5 \\ 2 & 3 & 3 \\ 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} e_{11}^{**} \\ e_{12}^{**} \\ e_{13}^{**} \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

giving $\mathbf{e}_1^{**} = [2 \;\; -1 \;\; 0]^T$.

$$\mathbf{M} = [\mathbf{e}_1 \; \mathbf{e}_1^* \; \mathbf{e}_1^{**}] = \begin{bmatrix} -3 & -1 & 2 \\ 1 & 1 & -1 \\ 1 & 0 & 0 \end{bmatrix}$$

$$\det \mathbf{M} = -1 \text{ and } \mathbf{M}^{-1} = -\begin{bmatrix} 0 & 0 & -1 \\ -1 & -2 & -1 \\ -1 & -1 & -2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$$

Matrix multiplication then confirms

$$\mathbf{M}^{-1}\,\mathbf{A}\,\mathbf{M} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

■ **27** Eigenvalues are given by

$$|\,\mathbf{A} - \lambda\mathbf{I}\,| = 0$$

that is, $\lambda^4 - 4\lambda^3 - 12\lambda^2 + 32\lambda + 64 = (\lambda + 2)^2(\lambda - 4)^2 = 0$ so the eigenvalues are -2, -2, 4 and 4 as required.

Corresponding to the repeated eigenvalue $\lambda_1, \lambda_2 = -2$

$$(\mathbf{A} + 2\mathbf{I}) = \begin{bmatrix} 3 & 0 & 0 & -3 \\ 0 & 3 & -3 & 0 \\ -0.5 & -3 & 3 & 0.5 \\ -3 & 0 & 0 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{ is of rank 2}$$

Thus, nullity $(\mathbf{A}+2\mathbf{I})$ is $4-2=2$ so there are two linearly independent eigenvectors corresponding to $\lambda = -2$.

Corresponding to the repeated eigenvalues $\lambda_3, \lambda_4 = 4$

$$(\mathbf{A} - 4\mathbf{I}) = \begin{bmatrix} -3 & 0 & 0 & -3 \\ 0 & -3 & -3 & 0 \\ -0.5 & -3 & -3 & 0.5 \\ -3 & 0 & 0 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \text{ is of rank 3}$$

Thus, nullity $(\mathbf{A} - 4\mathbf{I})$ is $4 - 3 = 1$ so there is only one linearly independent eigenvector corresponding to $\lambda = 4$.

When $\lambda = \lambda_1 = \lambda_2 = -2$ the eigenvalues are given by the solution of $(\mathbf{A}+2\mathbf{I})\mathbf{e} = 0$ giving $\mathbf{e}_1 = [0\ 1\ 1\ 0]^T, \mathbf{e}_2 = [1\ 0\ 0\ 1]^T$ as two linearly independent solutions. When $\lambda = \lambda_3 = \lambda_4 = 8$ the eigenvectors are given by the solution of

$$(\mathbf{A} - 4\mathbf{I})\mathbf{e} = 0$$

giving the unique solution $\mathbf{e}_3 = [0\ 1\ -1\ 0]^T$. The generalised eigenvector $\mathbf{e}_3^*$ is obtained by solving

$$(\mathbf{A} - 4\mathbf{I})\mathbf{e}_3^* = \mathbf{e}_3$$

giving $\mathbf{e}_3^* = (6\ \ -1\ \ \ 0\ \ -6]^T$. The Jordan canonical form is

$$\mathbf{J} = \left[\begin{array}{cc|cc} -2 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ \hline 0 & 0 & 4 & 1 \\ 0 & 0 & 0 & 4 \end{array}\right]$$

Exercises 1.6.5

■ **28** The quadratic form may be written in the form $V = \mathbf{x}^T\mathbf{A}\mathbf{x}$ where $\mathbf{x} = [x_1\ \ x_2\ \ x_3]^T$ and

$$\mathbf{A} = \begin{bmatrix} 2 & 2 & 1 \\ 2 & 5 & 2 \\ 1 & 2 & 2 \end{bmatrix}$$

The eigenvalues of $\boldsymbol{A}$ are given by

$$\begin{vmatrix} 2-\lambda & 2 & 1 \\ 2 & 5-\lambda & 2 \\ 1 & 2 & 2-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (2-\lambda)(\lambda^2 - 7\lambda + 6) + 4(\lambda - 1) + (\lambda - 1) = 0$$

$$\Rightarrow (\lambda - 1)(\lambda^2 - 8\lambda + 7) = 0 \Rightarrow (\lambda - 1)^2(\lambda - 7) = 0$$

giving the eigenvalues as $\lambda_1 = 7, \lambda_2 = \lambda_3 = 1$
Normalized eigenvector corresponding to $\lambda_1 = 7$ is

$$\hat{\mathbf{e}}_1 = \left[\ \tfrac{1}{\sqrt{6}}\ \ \ \tfrac{2}{\sqrt{6}}\ \ \ \tfrac{1}{\sqrt{6}}\ \right]^T$$

and two orthogonal linearly independent eigenvectors corresponding to $\lambda - 1$ are

$$\hat{\mathbf{e}}_2 = \left[\tfrac{1}{\sqrt{2}} \quad 0 \quad -\tfrac{1}{\sqrt{2}}\right]^T$$
$$\hat{\mathbf{e}}_3 = \left[-\tfrac{1}{\sqrt{3}} \quad \tfrac{1}{\sqrt{3}} \quad -\tfrac{1}{\sqrt{3}}\right]^T$$

Note that $\hat{\mathbf{e}}_2$ and $\hat{\mathbf{e}}_3$ are automatically orthogonal to $\hat{\mathbf{e}}_1$. The normalized orthogonal modal matrix $\hat{\mathbf{M}}$ and spectral matrix $\mathbf{\Lambda}$ are

$$\hat{\mathbf{M}} = \begin{bmatrix} \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{3}} \\ \frac{2}{\sqrt{6}} & 0 & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{3}} \end{bmatrix}, \mathbf{\Lambda} = \begin{bmatrix} 7 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

such that $\hat{\mathbf{M}}^T\mathbf{A}\hat{\mathbf{M}} = \mathbf{\Lambda}$.
Under the orthogonal transformation $\mathbf{x} = \hat{\mathbf{M}}\mathbf{y}$ the quadratic form V reduces to

$$\begin{aligned} V &= \mathbf{y}^T\hat{\mathbf{M}}^T\mathbf{A}\hat{\mathbf{M}}\mathbf{y} = \mathbf{y}^T\mathbf{\Lambda}\mathbf{y} \\ &= [y_1 \quad y_2 \quad y_3] \begin{bmatrix} 7 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \\ &= 7y_1^2 + y_2^2 + y_3^2 \end{aligned}$$

- **29(a)** The matrix of the quadratic form is $\mathbf{A} = \begin{bmatrix} 1 & -1 & 2 \\ -1 & 2 & -1 \\ 2 & -1 & 7 \end{bmatrix}$ and its leading principal minors are

$$1, \begin{vmatrix} 1 & -1 \\ -1 & 2 \end{vmatrix} = 1, \det \mathbf{A} = 2$$

Thus, by Sylvester's condition (a) the quadratic form is positive definite.

29(b) Matrix $\mathbf{A} = \begin{bmatrix} 1 & -1 & 2 \\ -1 & 2 & -1 \\ 2 & -1 & 5 \end{bmatrix}$ and its leading principal minors are

$$1, \begin{vmatrix} 1 & -1 \\ -1 & 2 \end{vmatrix} = 1, \det \mathbf{A} = 0$$

Thus, by Sylvester's condition (c) the quadratic form is positive semidefinite.

29(c) Matrix $\mathbf{A} = \begin{bmatrix} 1 & -1 & 2 \\ -1 & 2 & -1 \\ 2 & -1 & 4 \end{bmatrix}$ and its leading principal minors are

$$1, \begin{vmatrix} 1 & -1 \\ -1 & 2 \end{vmatrix} = 1, \det \mathbf{A} = -1.$$

Thus, none of Sylvester's conditions are satisfied and the quadratic form is indefinite.

▪ **30(a)** The matrix of the quadratic form is $\mathbf{A} = \begin{bmatrix} a & -b \\ -b & c \end{bmatrix}$ and its leading principal minors are a and $ac - b^2$. By Sylvester's condition (a) in the text the quadratic form is positive definite if and only if

$$a > 0 \text{ and } ac - b^2 > 0$$
$$\text{that is, } a > 0 \text{ and } ac > b^2$$

30(b) The matrix of the quadratic form is $\mathbf{A} = \begin{bmatrix} 2 & -1 & 0 \\ -1 & a & b \\ 0 & b & 3 \end{bmatrix}$ having principal minors $2, 2a - 1$ and $\det \mathbf{A} = 6a - 2b^2 - 3$. Thus, by Sylvester's condition (a) in the text the quadratic form is positive definite if and only if

$$2a - 1 > 0 \text{ and } 6a - 2b^2 - 3 > 0$$
$$\text{or } 2a > 1 \text{ and } 2b^2 < 6a - 3$$

▪ **31** The eigenvalues of the matrix $\mathbf{A}$ are given by

$$0 = \begin{vmatrix} 2-\lambda & 1 & -1 \\ 1 & 2-\lambda & 1 \\ -1 & 1 & 2-\lambda \end{vmatrix} \underset{=}{\mathrm{R}_1 + \mathrm{R}_3} \begin{vmatrix} 3-\lambda & 3-\lambda & 0 \\ 1 & 2-\lambda & 1 \\ -1 & 1 & 2-\lambda \end{vmatrix}$$

$$= (3-\lambda) \begin{vmatrix} 1 & 1 & 0 \\ 1 & 2-\lambda & 1 \\ -1 & 1 & 2-\lambda \end{vmatrix}$$

$$= (3-\lambda) \begin{vmatrix} 1 & 0 & 0 \\ 1 & 1-\lambda & 1 \\ -1 & 2 & 2-\lambda \end{vmatrix} = (3-\lambda)(\lambda^2 - 3\lambda)$$

so the eigenvalues are $3, 3, 0$ indicating that the matrix is positive semidefinite. The principal minors of $\mathbf{A}$ are

$$2, \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} = 3, \ \det \mathbf{A} = 0$$

confirming, by Sylvester's condition (a), that the matrix is positive semidefinite.

■ **32** The matrix of the quadratic form is $\mathbf{A} = \begin{bmatrix} K & 1 & 1 \\ 1 & K & -1 \\ 1 & -1 & 1 \end{bmatrix}$ having principal minors

$$K, \begin{vmatrix} K & 1 \\ 1 & K \end{vmatrix} = K^2 - 1 \text{ and } \det \mathbf{A} = K^2 - K - 3$$

Thus, by Sylvester's condition (a) the quadratic form is positive definite if and only if

$$K^2 - 1 = (K-1)(K+1) > 0 \text{ and } K^2 - K - 3 = (K-2)(K+1) > 0$$

i.e. $K > 2$ or $K < -1$.

If $K = 2$ then $\det \mathbf{A} = 0$ and the quadratic form is positive semidefinite.

■ **33** Principal minors of the matrix are

$$3 + a, \begin{vmatrix} 3 + a & 1 \\ 1 & a \end{vmatrix} = a^2 + 3a - 1, \det \mathbf{A} = a^3 + 3a^2 - 6a - 8$$

Thus, by Sylvester's condition (a) the quadratic form is positive definite if and only if

$$3 + a > 0, a^2 + 3a - 1 > 0 \text{ and } a^3 + 3a^2 - 6a - 8 > 0$$
$$\text{or } (a+1)(a+4)(a-2) > 0$$

$$3 + a > 0 \ \Rightarrow \ a > -3$$
$$a^2 + 3a - 1 > 0 \ \Rightarrow \ a < -3.3 \text{ or } a > 0.3$$
$$(a+1)(a+4)(a-2) > 0 \ \Rightarrow \ a > 2 \text{ or } -4 < a < -1$$

Thus, minimum value of a for which the quadratic form is positive definite is $a = 2$.

■ **34** $\mathbf{A} = \begin{bmatrix} 1 & 2 & -2 \\ 2 & \lambda & -3 \\ -2 & -3 & \lambda \end{bmatrix}$

Principal minors are

$$1, \begin{vmatrix} 1 & 2 \\ 2 & \lambda \end{vmatrix} = \lambda - 4, \ \det \mathbf{A} = \lambda^2 - 8\lambda + 15 = 0$$

Thus, by Sylvester's condition (a) the quadratic form is positive definite if and only if

$$\lambda - 4 > 0 \ \Rightarrow \ \lambda > 4$$

$$\text{and } (\lambda - 5)(\lambda - 3) > 0 \ \Rightarrow \ \lambda < 3 \text{ or } \lambda > 5$$

Thus, it is positive definite if and only if $\lambda > 5$.

Exercises 1.7.1

■ **35** The characteristic equation of $\mathbf{A}$ is

$$\begin{vmatrix} 5-\lambda & 6 \\ 2 & 3-\lambda \end{vmatrix} = \lambda^2 - 8\lambda + 3 = 0$$

$$\text{Now } \mathbf{A}^2 = \begin{bmatrix} 5 & 6 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 5 & 6 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 27 & 48 \\ 16 & 21 \end{bmatrix} \text{ so}$$

$$\mathbf{A}^2 - 8\mathbf{A} + 3\mathbf{I} = \begin{bmatrix} 37 & 48 \\ 16 & 21 \end{bmatrix} - \begin{bmatrix} 40 & 48 \\ 16 & 24 \end{bmatrix} + \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

so that $\mathbf{A}$ satisfies its own characteristic equation.

■ **36** The characteristic equation of $\mathbf{A}$ is

$$\begin{vmatrix} 1-\lambda & 2 \\ 1 & 1-\lambda \end{vmatrix} = \lambda^2 - 2\lambda - 1 = 0$$

By Cayley–Hamilton theorem

$$\mathbf{A}^2 - 2\mathbf{A} - \mathbf{I} = 0$$

36(a) Follows that $\mathbf{A}^2 = 2\mathbf{A} + \mathbf{I} = \begin{bmatrix} 2 & 4 \\ 2 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ 2 & 3 \end{bmatrix}$

36(b) $\mathbf{A}^3 = 2\mathbf{A}^2 + \mathbf{A} = \begin{bmatrix} 6 & 8 \\ 4 & 6 \end{bmatrix} + \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 7 & 10 \\ 5 & 7 \end{bmatrix}$

36(c) $\mathbf{A}^4 = 2\mathbf{A}^3 + \mathbf{A}^2 = \begin{bmatrix} 14 & 20 \\ 10 & 14 \end{bmatrix} + \begin{bmatrix} 3 & 4 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 17 & 24 \\ 12 & 17 \end{bmatrix}$

▪ **37(a)** The characteristic equation of **A** is

$$\begin{vmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{vmatrix} = 0$$

$$\text{that is,} \quad \lambda^2 - 4\lambda + 3 = 0$$

Thus, by the Cayley–Hamilton theorem

$$\begin{aligned} \mathbf{A}^2 - 4\mathbf{A} + 3\mathbf{I} &= 0 \\ \mathbf{I} &= \frac{1}{3}[4\mathbf{A} - \mathbf{A}^2] \\ \text{so that} \quad \mathbf{A}^{-1} &= \frac{1}{3}[4\mathbf{I} - \mathbf{A}] \\ &= \frac{1}{3}\left\{\begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} - \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\right\} = \frac{1}{3}\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \end{aligned}$$

37(b) The characteristic equation of **A** is

$$\begin{vmatrix} 1-\lambda & 1 & 2 \\ 3 & 1-\lambda & 1 \\ 2 & 3 & 1-\lambda \end{vmatrix} = 0$$

$$\text{that is,} \quad \lambda^3 - 3\lambda^2 - 7\lambda - 11 = 0$$

$$\mathbf{A}^2 = \begin{bmatrix} 1 & 1 & 2 \\ 3 & 1 & 1 \\ 2 & 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 2 \\ 3 & 1 & 1 \\ 2 & 3 & 1 \end{bmatrix} = \begin{bmatrix} 8 & 8 & 5 \\ 8 & 7 & 8 \\ 13 & 8 & 8 \end{bmatrix}$$

Using (1.44)

$$\mathbf{A}^{-1} = \frac{1}{11}(\mathbf{A}^2 - 3\mathbf{A} - 7\mathbf{I})$$
$$= \frac{1}{11} \begin{bmatrix} -2 & 5 & -1 \\ -1 & -3 & 5 \\ 7 & -1 & -2 \end{bmatrix}$$

■ **38** $\mathbf{A}^2 = \begin{bmatrix} 2 & 3 & 1 \\ 3 & 1 & 2 \\ 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 2 & 3 & 1 \\ 3 & 1 & 2 \\ 1 & 2 & 3 \end{bmatrix} = \begin{bmatrix} 14 & 11 & 11 \\ 11 & 14 & 11 \\ 11 & 11 & 14 \end{bmatrix}$

The characteristic equation of $\mathbf{A}$ is

$$\lambda^2 - 6\lambda^2 - 3\lambda + 18 = 0$$

so by the Cayley–Hamilton theorem

$$\mathbf{A}^3 = 6\mathbf{A}^2 + 3\mathbf{A} - 18\mathbf{I}$$

giving

$$\mathbf{A}^4 = 6(6\mathbf{A}^2 + 3\mathbf{A} - 18\mathbf{I}) + 3\mathbf{A}^2 - 18\mathbf{A} = 39\mathbf{A}^2 - 108\mathbf{I}$$
$$\mathbf{A}^5 = 39(6\mathbf{A}^2 + 3\mathbf{A} - 18\mathbf{I}) + 108\mathbf{A} = 234\mathbf{A}^2 + 9\mathbf{A} - 702\mathbf{I}$$
$$\mathbf{A}^6 = 234(6\mathbf{A}^2 + 3\mathbf{A} - 18\mathbf{I}) + 9\mathbf{A}^2 - 702\mathbf{A} = 1413\mathbf{A}^2 - 4212\mathbf{I}$$
$$\mathbf{A}^7 = 1413(6\mathbf{A}^2 + 3\mathbf{A} - 18\mathbf{I}) + 4212\mathbf{A} = 8478\mathbf{A}^2 + 27\mathbf{A} - 25434\mathbf{I}$$

Thus,

$$\mathbf{A}^7 - 3\mathbf{A}^6 + \mathbf{A}^4 + 3\mathbf{A}^3 - 2\mathbf{A}^2 + 3\mathbf{I} = 4294\mathbf{A}^2 + 36\mathbf{A} - 12957\mathbf{I}$$
$$= \begin{bmatrix} 47231 & 47342 & 47270 \\ 47342 & 47195 & 47306 \\ 47270 & 47306 & 47267 \end{bmatrix}$$

▪ **39(a)** Eigenvalues **A** are $\lambda = 1$ (repeated). Thus,

$$e^{\mathbf{A}t} = \alpha_0\mathbf{I} + \alpha_1\mathbf{A} \text{ with}$$

$$\left.\begin{array}{ll} e^t & = \alpha_0 + \alpha_1 \\ te^t & = \alpha_1 \end{array}\right\} \Rightarrow \alpha_1 = te^t,\ \alpha_0 = (1-t)e^t$$

$$\text{so } e^{\mathbf{A}t} = (1-t)e^t\mathbf{I} + te^t\mathbf{A} = \begin{bmatrix} e^t & 0 \\ te^t & e^t \end{bmatrix}$$

39(b) Eigenvalues **A** are $\lambda = 1$ and $\lambda = 2$. Thus,

$$e^{\mathbf{A}t} = \alpha_0\mathbf{I} + \alpha_1\mathbf{A} \text{ with}$$

$$\left.\begin{array}{ll} e^t & = \alpha_0 + \alpha_1 \\ e^{2t} & = \alpha_0 + 2\alpha_1 \end{array}\right\} \Rightarrow \alpha_0 = 2e^t - e^{2t},\ \alpha_1 = e^{2t} - e^t$$

$$\text{so } e^{\mathbf{A}t} = (2e^t - e^{2t})\mathbf{I} + (e^{2t} - e^t)\mathbf{A} = \begin{bmatrix} e^t & 0 \\ e^{2t} - e^t & e^{2t} \end{bmatrix}$$

▪ **40** Eigenvalues of **A** are $\lambda_1 = \pi, \lambda_2 = \frac{\pi}{2}, \lambda_3 = \frac{\pi}{2}$.
Thus,

$$\sin\mathbf{A} = \alpha_0\mathbf{A} + \alpha_1\mathbf{A} + \alpha_2\mathbf{A}^2 \text{ with}$$

$$\sin\pi = 0 = \alpha_0 + \alpha_1\pi + \alpha_2\pi^2$$

$$\sin\frac{\pi}{2} = 1 = \alpha_0 + \alpha_1\frac{\pi}{2} + \alpha_2\frac{\pi^2}{4}$$

$$\cos\frac{\pi}{2} = 0 = \alpha_1 + \pi\alpha_2$$

Solving gives $\alpha_0 = 0,\ \alpha_1 = \frac{4}{\pi},\ \alpha_2 = -\frac{4}{\pi^2}$ so that

$$\sin\mathbf{A} = \frac{4}{\pi}\mathbf{A} - \frac{4}{\pi^2}\mathbf{A}^2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

▪ **41(a)**

$$\frac{d\mathbf{A}}{dt} = \begin{bmatrix} \frac{d}{dt}(t^2+1) & \frac{d}{dt}(2t-3) \\ \frac{d}{dt}(5-t) & \frac{d}{dt}(t^2-t+3) \end{bmatrix} = \begin{bmatrix} 2t & 2 \\ -1 & 2t-1 \end{bmatrix}$$

41(b)

$$\int_1^2 \mathbf{A}dt = \begin{bmatrix} \int_1^2(t^2+1)dt & \int_1^2(2t-3)dt \\ \int_1^2(5-t)dt & \int_1^2(t^2-t+3)dt \end{bmatrix} = \begin{bmatrix} \frac{10}{3} & 0 \\ \frac{7}{2} & \frac{23}{6} \end{bmatrix}$$

42

$$\begin{aligned} \mathbf{A}^2 &= \begin{bmatrix} t^2+1 & t-1 \\ 5 & 0 \end{bmatrix} \begin{bmatrix} t^2+1 & t-1 \\ 5 & 0 \end{bmatrix} \\ &= \begin{bmatrix} t^4+2t^2+5t-4 & t^3-t^2+t-1 \\ 5t^2+5 & 5t-5 \end{bmatrix} \\ \frac{d}{dt}(\mathbf{A}^2) &= \begin{bmatrix} 4t^3+4t+5 & 3t^2-2t+1 \\ 10t & 5 \end{bmatrix} \\ 2\mathbf{A}\frac{d\mathbf{A}}{dt} &= \begin{bmatrix} 4t^3+4t & 2t^2+1 \\ 20t & 0 \end{bmatrix} \end{aligned}$$

Thus, $\frac{d}{dt}(\mathbf{A}^2) \neq 2\mathbf{A}\frac{d\mathbf{A}}{dt}$.

Exercises 1.8.4

43(a) *row rank*

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 7 & 10 \\ 2 & 1 & 5 & 7 \end{bmatrix} \begin{matrix} \text{row2} - 3\text{row1} \\ \rightarrow \\ \text{row3} - 2\text{row1} \end{matrix} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -2 & -2 & -2 \\ 0 & -3 & -1 & -1 \end{bmatrix}$$

$$\begin{matrix} -\frac{1}{2}\text{row2} \\ \rightarrow \end{matrix} \begin{bmatrix} 1 & 2 & 4 & 4 \\ 0 & 1 & 1 & 1 \\ 0 & -3 & -1 & -1 \end{bmatrix} \begin{matrix} \text{row3} + 3\text{row2} \\ \rightarrow \end{matrix} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 2 & 2 \end{bmatrix}$$

echelon form, row rank 3

column rank

$$\mathbf{A} \begin{matrix} \text{col2} - 2\text{col1} \\ \rightarrow \\ \text{col3} - 3\text{col1} \\ \text{col4} - 4\text{col1} \end{matrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 3 & -2 & -2 & 2 \\ 2 & -3 & 2 & 0 \end{bmatrix} \begin{matrix} \text{col3} - \text{col2} \\ \rightarrow \\ \text{col4} - \text{col2} \end{matrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 2 & -3 & 2 & 2 \end{bmatrix}$$

$$\begin{matrix} \text{col4} - \text{col3} \\ \rightarrow \end{matrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 3 & -2 & 0 & 0 \\ 2 & -3 & 2 & 0 \end{bmatrix}$$

echelon form, column rank3

Thus row rank($\mathbf{A}$) = column rank($\mathbf{A}$) = 3

(b) $\mathbf{A}$ is of full rank since rank($\mathbf{A}$)=min(m, n)=min(3,4)= 3

■ **44(a)** $\mathbf{AA}^T = \begin{bmatrix} 4 & 11 & 14 \\ 8 & 7 & -2 \end{bmatrix} \begin{bmatrix} 4 & 8 \\ 11 & 7 \\ 14 & -2 \end{bmatrix} = \begin{bmatrix} 333 & 81 \\ 81 & 117 \end{bmatrix} = 9 \begin{bmatrix} 37 & 9 \\ 9 & 13 \end{bmatrix}$

The eigenvalues λ_i of $\boldsymbol{AA}^T$ are given by the solutions of the equations

$$\left|\mathbf{AA}^T - \lambda \mathbf{I}\right| = \begin{vmatrix} 333-\lambda & 8 \\ 81 & 117-\lambda \end{vmatrix} = 0 \Rightarrow \lambda^2 - 450\lambda + 32400 = 0$$

$$\Rightarrow (\lambda - 360)(\lambda - 90) = 0$$

giving the eigenvalues as $\lambda_1 = 360, \lambda_2 = 90$. Solving the equations.

$$(\mathbf{AA}^T - \lambda_i \mathbf{I})\mathbf{u}_i = 0$$

gives the corresponding eigenvectors as

$$\mathbf{u}_1 = [3 \quad 1]^T, \mathbf{u}_2 = [1 \quad -2]^T$$

with the corresponding normalized eigenvectors being

$$\hat{\mathbf{u}}_1 = [\tfrac{3}{\sqrt{10}} \quad \tfrac{1}{\sqrt{10}}]^T, \hat{\mathbf{u}}_2 = [\tfrac{1}{\sqrt{10}} \quad -\tfrac{3}{\sqrt{10}}]^T$$

leading to the orthogonal matrix

$$\hat{\mathbf{U}} = \begin{bmatrix} \frac{3}{\sqrt{10}} & \frac{1}{\sqrt{10}} \\ \frac{1}{\sqrt{10}} & -\frac{3}{\sqrt{10}} \end{bmatrix}$$

$$\mathbf{A}^T\mathbf{A} = \begin{bmatrix} 4 & 8 \\ 11 & 7 \\ 14 & -2 \end{bmatrix} \begin{bmatrix} 4 & 11 & 14 \\ 8 & 7 & -2 \end{bmatrix} = \begin{bmatrix} 80 & 100 & 40 \\ 100 & 170 & 140 \\ 40 & 140 & 200 \end{bmatrix}$$

Solving $\left|\mathbf{A}^T\mathbf{A} - \mu \mathbf{I}\right| = \begin{vmatrix} 80-\mu & 100 & 40 \\ 100 & 170-\mu & 140 \\ 40 & 140 & 200-\mu \end{vmatrix} = 0$

gives the eigenvalues $\mu_1 = 360, \mu_2 = 90, \mu_3 = 0$ with corresponding normalized eigenvectors

$$\hat{\mathbf{v}}_1 = [\tfrac{1}{3} \quad \tfrac{2}{3} \quad \tfrac{2}{3}]^T, \hat{\mathbf{v}}_2 = [-\tfrac{2}{3} \quad -\tfrac{1}{3} \quad \tfrac{2}{3}]^T, \hat{\mathbf{v}}_3 = [\tfrac{2}{3} \quad -\tfrac{2}{3} \quad \tfrac{1}{3}]^T$$

leading to the orthogonal matrix

$$\hat{\mathbf{V}} = \begin{bmatrix} \frac{1}{3} & -\frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & -\frac{1}{3} & -\frac{2}{3} \\ \frac{2}{3} & \frac{2}{3} & \frac{1}{3} \end{bmatrix}$$

The singular values of $\mathbf{A}$ are $\sigma_1 = \sqrt{360} = 6\sqrt{10}$ and $\sigma_2 = \sqrt{90} = 3\sqrt{10}$ giving

$$\mathbf{\Sigma} = \begin{bmatrix} 6\sqrt{10} & 0 & 0 \\ 0 & 3\sqrt{10} & 0 \end{bmatrix}$$

Thus, the SVD form of $\mathbf{A}$ is

$$\mathbf{A} = \hat{\mathbf{U}}\mathbf{\Sigma}\hat{\mathbf{V}}^T = \begin{bmatrix} \frac{3}{\sqrt{10}} & \frac{1}{\sqrt{10}} \\ \frac{1}{\sqrt{10}} & -\frac{3}{\sqrt{10}} \end{bmatrix} \begin{bmatrix} 6\sqrt{10} & 0 & 0 \\ 0 & 3\sqrt{10} & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ -\frac{2}{3} & -\frac{1}{3} & \frac{2}{3} \\ \frac{2}{3} & -\frac{2}{3} & \frac{1}{3} \end{bmatrix}$$

(Direct multiplication confirms $\mathbf{A} = \begin{bmatrix} 4 & 11 & 14 \\ 8 & 7 & -2 \end{bmatrix}$)

(b) Using (1.55) the pseudo inverse of $\boldsymbol{A}$ is

$$\mathbf{A}^\dagger = \hat{\mathbf{V}}\mathbf{\Sigma}^*\hat{\mathbf{U}},\ \mathbf{\Sigma}^* = \begin{bmatrix} \frac{1}{6\sqrt{10}} & 0 \\ 0 & \frac{2}{3\sqrt{10}} \\ 0 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} \frac{1}{3} & -\frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & -\frac{1}{3} & -\frac{2}{3} \\ \frac{2}{3} & \frac{2}{3} & \frac{1}{3} \end{bmatrix} \begin{bmatrix} \frac{1}{6\sqrt{10}} & 0 \\ 0 & \frac{1}{3\sqrt{10}} \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} \frac{3}{\sqrt{10}} & \frac{1}{\sqrt{10}} \\ \frac{1}{\sqrt{10}} & -\frac{3}{\sqrt{10}} \end{bmatrix} \Rightarrow \mathbf{A}^\dagger = \tfrac{1}{180} \begin{bmatrix} -1 & 13 \\ 4 & 8 \\ 10 & -10 \end{bmatrix}$$

$$\mathbf{A}\mathbf{A}^\dagger = \tfrac{1}{180} \begin{bmatrix} 4 & 11 & 14 \\ 8 & 7 & -2 \end{bmatrix} \begin{bmatrix} -1 & 13 \\ 4 & 8 \\ 10 & -10 \end{bmatrix} = \tfrac{1}{180} \begin{bmatrix} 180 & 0 \\ 0 & 180 \end{bmatrix} = \mathbf{I}$$

(c) Rank$(\boldsymbol{A}) = 2$ so $\boldsymbol{A}$ is of full rank. Since number of rows is less than the number of columns $\mathbf{A}^\dagger$ may be determined using (1.58b) as

$$\mathbf{A}^\dagger = \mathbf{A}^T(\mathbf{A}\mathbf{A}^T)^{-1} = \begin{bmatrix} 4 & 8 \\ 11 & 7 \\ 14 & -2 \end{bmatrix} \begin{bmatrix} 333 & 81 \\ 81 & 117 \end{bmatrix}^{-1} = \tfrac{1}{180} \begin{bmatrix} -1 & 13 \\ 4 & 8 \\ 10 & -10 \end{bmatrix}$$

which confirms with the value determined in (b).

■ **45** $\mathbf{A} = \begin{bmatrix} 1 & 1 \\ 3 & 0 \\ -2 & 1 \\ 0 & 2 \\ -1 & 2 \end{bmatrix} \begin{matrix} \text{row2} - 3\text{row1} \\ \text{row3} + 2\text{row1} \\ \rightarrow \\ \text{row5} + \text{row1} \end{matrix} \begin{bmatrix} 1 & 1 \\ 0 & -3 \\ 0 & 3 \\ 0 & 2 \\ 0 & 3 \end{bmatrix} \begin{matrix} \text{row3} + \text{row2} \\ \text{row4} + \frac{2}{3}\text{row2} \\ \rightarrow \\ \text{row5} + \text{row2} \end{matrix} \begin{bmatrix} 1 & 1 \\ 0 & -3 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$

echelon form so row rank = 2 = column rank

Thus, rank $\boldsymbol{A}$ = 2 =min(5,2) and so $\boldsymbol{A}$ is of full rank.

Since $\boldsymbol{A}$ is of full rank and number of rows is greater than number of columns we can determine the pseudo inverse using result (1.58a)

$$\mathbf{A}^\dagger = (\mathbf{A}^T\mathbf{A})^{-1}\mathbf{A}^T = \begin{bmatrix} 15 & -3 \\ -3 & 10 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 3 & -2 & 0 & -1 \\ 1 & 0 & 1 & 2 & 2 \end{bmatrix}$$
$$= \tfrac{1}{141} \begin{bmatrix} 10 & 3 \\ 3 & 15 \end{bmatrix} \begin{bmatrix} 1 & 3 & -2 & 0 & -1 \\ 1 & 0 & 1 & 2 & 2 \end{bmatrix}$$
$$= \tfrac{1}{141} \begin{bmatrix} 13 & 30 & -17 & 6 & -4 \\ 18 & 9 & 9 & 30 & 27 \end{bmatrix}$$

$$\mathbf{A}^\dagger\mathbf{A} = \tfrac{1}{141} \begin{bmatrix} 13 & 30 & -17 & 6 & -4 \\ 18 & 9 & 9 & 30 & 27 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 3 & 0 \\ -2 & 1 \\ 0 & 2 \\ -1 & 2 \end{bmatrix} = \tfrac{1}{141} \begin{bmatrix} 141 & 0 \\ 0 & 141 \end{bmatrix} = \mathbf{I}$$

■ **46(a)** $\mathbf{A} = \begin{bmatrix} 1 & -1 \\ -2 & 2 \\ 2 & -2 \end{bmatrix} \begin{matrix} \text{row2} + 2\text{row1} \\ \rightarrow \\ \text{row3} - 2\text{row1} \end{matrix} \begin{bmatrix} 1 & -1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$

Thus, rank $\mathbf{A}$ = 1and is not of full rank

(b) $\mathbf{A}\mathbf{A}^T = \begin{bmatrix} 1 & -1 \\ -2 & 2 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} 1 & 2 & 2 \\ -1 & 2 & -2 \end{bmatrix} = \begin{bmatrix} 2 & -4 & 4 \\ -4 & 8 & -8 \\ 4 & -8 & 8 \end{bmatrix}$

The eigenvalues λ_i are given by

$$\begin{vmatrix} 2-\lambda & -4 & 4 \\ -4 & 2-\lambda & -8 \\ 4 & -8 & 8-\lambda \end{vmatrix} = 0 \Rightarrow \lambda^2(-\lambda + 18) = 0$$

giving the eigenvalues as $\lambda_1 = 18, \lambda_2 = 0, \lambda_3 = 0$. The corresponding eigenvectors and normalized eigenvectors are

$$\mathbf{u}_1 = [1 \quad -2 \quad 2]^T \Rightarrow \hat{\mathbf{u}}_1 = [\tfrac{1}{3} \quad -\tfrac{2}{3} \quad \tfrac{2}{3}]^T$$
$$\mathbf{u}_2 = [0 \quad 1 \quad 1]^T \Rightarrow \hat{\mathbf{u}}_2 = [0 \quad \tfrac{1}{\sqrt{2}} \quad \tfrac{1}{\sqrt{2}}]^T$$
$$\mathbf{u}_3 = [2 \quad 1 \quad 0]^T \Rightarrow \hat{\mathbf{u}}_3 = [\tfrac{2}{\sqrt{5}} \quad \tfrac{1}{\sqrt{5}} \quad 0]^T$$

leading to the orthogonal matrix

$$\hat{\mathbf{U}} = \begin{bmatrix} \frac{1}{3} & 0 & \frac{2}{\sqrt{5}} \\ -\frac{2}{3} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{5}} \\ \frac{2}{3} & \frac{1}{\sqrt{2}} & 0 \end{bmatrix}$$

$$\mathbf{A}^T\mathbf{A} = \begin{bmatrix} 1 & -2 & 2 \\ -1 & 2 & -2 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -2 & 2 \\ 2 & -2 \end{bmatrix} = 9 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

having eigenvalues $\mu_1 = 18$ and $\mu_2 = 0$ and corresponding eigenvectors

$$\mathbf{v}_1 = [1 \quad -1]^T \Rightarrow \hat{\mathbf{v}}_1 = [\tfrac{1}{\sqrt{2}} \quad -\tfrac{1}{\sqrt{2}}]^T$$
$$\mathbf{v}_2 = [1 \quad 1]^T \Rightarrow \hat{\mathbf{v}}_2 = [\tfrac{1}{\sqrt{2}} \quad \tfrac{1}{\sqrt{2}}]^T$$

leading to the orthogonal matrix

$$\hat{\mathbf{V}} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

$\boldsymbol{A}$ has the single (equal to its rank) singular value $\sigma_1 = \sqrt{18} = 3\sqrt{2}$ so that $\mathbf{\Sigma} = \begin{bmatrix} 3\sqrt{2} & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$ and the SVD form of $\boldsymbol{A}$ is

$$\mathbf{A} = \hat{\mathbf{U}}\mathbf{\Sigma}\hat{\mathbf{V}}^T = \begin{bmatrix} \frac{1}{3} & 0 & \frac{2}{\sqrt{5}} \\ -\frac{2}{3} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{5}} \\ \frac{2}{3} & \frac{1}{\sqrt{2}} & 0 \end{bmatrix} \begin{bmatrix} 3\sqrt{2} & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

Direct multiplication confirms that $\mathbf{A} = \begin{bmatrix} 1 & -1 \\ -2 & 2 \\ 2 & -2 \end{bmatrix}$

(c) Pseudo inverse is given by

$$\mathbf{A}^\dagger = \hat{\mathbf{V}}\mathbf{\Sigma}^*\hat{\mathbf{U}}^T = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \frac{1}{3\sqrt{2}} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{3} & -\frac{2}{3} & \frac{2}{3} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} & 0 \end{bmatrix} = \frac{1}{18} \begin{bmatrix} 1 & -2 & 2 \\ -1 & 2 & -2 \end{bmatrix}$$

Direct multiplication confirms $\mathbf{A}\mathbf{A}^{\dagger}\mathbf{A} = \mathbf{A}and\mathbf{A}^{\dagger}\mathbf{A}\mathbf{A}^{\dagger} = \mathbf{A}^{\dagger}$

(d) Equations may be written as

$$\begin{bmatrix} 1 & -1 \\ -2 & 2 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \equiv \mathbf{A}\mathbf{x} = \mathbf{b}$$

The least squares solution is $\mathbf{x} = \mathbf{A}^{\dagger}\mathbf{b} \Rightarrow \begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{18}\begin{bmatrix} 1 & -2 & 2 \\ -1 & 2 & -2 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} =$ $\begin{bmatrix} \frac{1}{6} \\ -\frac{1}{6} \end{bmatrix}$ giving $x = \frac{1}{6}$ and $y = -\frac{1}{6}$

(e) Minimize $L = (x - y - 1)^2 + (-2x + 2y - 2)^2 + (2x - 2y - 3)^2$

$$\frac{\partial L}{\partial x} = 0 \Rightarrow 2(x - y - 1) - 4(-2x + 2y - 2) + 4(2x - 2y - 3) = 18x - 18y - 6 = 0$$
$$\Rightarrow 3x - 3y - 1 = 0$$
$$\frac{\partial L}{\partial y} = 0 \Rightarrow -2(x - y - 1) + 4(-2x + 2y - 2) - 4(2x - 2y - 3) = -18x + 18y + 6 = 0$$
$$\Rightarrow -3x + 3y + 1 = 0$$

Solving the two simultaneous equations gives the least squares solution $x = \frac{1}{6}$, $y = -\frac{1}{6}$ confirming the answer in (d)

- **47(a)** Equations may be written as

$$\begin{bmatrix} 3 & -1 \\ 1 & 3 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \equiv \mathbf{A}\mathbf{x} = \mathbf{b}$$

Using the pseudo inverse obtained in Example 1.39, the least squares solution is

$$\mathbf{x} = \mathbf{A}^{\dagger}\mathbf{b} \Rightarrow \begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{60}\begin{bmatrix} 17 & 4 & 5 \\ -7 & 16 & 5 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} \frac{2}{3} \\ \frac{2}{3} \end{bmatrix}$$

giving $x = y = \frac{2}{3}$

(b) Minimize $L = (3x - y - 1)^2 + (x + 3y - 2)^2 + (x + y - 3)^2$

$$\frac{\partial L}{\partial x} = 0 \Rightarrow 6(3x - y - 1) + 2(x + 3y - 2) + 2(x + y - 3) = 0$$
$$\Rightarrow 11x + y - 8 = 0$$
$$\frac{\partial L}{\partial y} = 0 \Rightarrow -2(3x - y - 1) + 6(x + 3y - 2) + 2(x + y - 3) = 0$$
$$\Rightarrow x + 11y - 8 = 0$$

Solving the two simultaneous equations gives the least squares solution $x = y = \frac{2}{3}$ confirming the answer in (a)

- **48(a)**

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & -1 \\ -1 & 1 & 1 \\ 2 & -1 & 2 \end{bmatrix} \begin{matrix} \text{row3 + row1} \\ \rightarrow \\ \text{row4} - \text{2row1} \end{matrix} \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & -1 \\ 0 & 1 & -1 \\ 0 & -1 & 6 \end{bmatrix} \begin{matrix} \text{row3} - \text{row2} \\ \rightarrow \\ \text{row4 + row2} \end{matrix} \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

Thus, $\boldsymbol{A}$ is of rank 3 and is of full rank as 3=min(4,3)

(b) Since $\boldsymbol{A}$ is of full rank

$$\mathbf{A}^\dagger = (\mathbf{A}^T\mathbf{A})^{-1}\mathbf{A}^T = \begin{bmatrix} 6 & -3 & 1 \\ -3 & 3 & -2 \\ 1 & -2 & 10 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 & -1 & 2 \\ 0 & 1 & 1 & -1 \\ -2 & -1 & 1 & 2 \end{bmatrix}$$

$$\Rightarrow \mathbf{A}^\dagger = \frac{1}{75} \begin{bmatrix} 26 & 28 & 3 \\ 28 & 59 & 9 \\ 3 & 9 & 9 \end{bmatrix} \begin{bmatrix} 1 & 0 & -1 & 2 \\ 0 & 1 & 1 & -1 \\ -2 & -1 & 1 & 2 \end{bmatrix} = \frac{1}{15} \begin{bmatrix} 4 & 5 & 1 & 6 \\ 2 & 10 & 8 & 3 \\ -3 & 0 & 3 & 3 \end{bmatrix}$$

(c) Direct multiplication confirms that $\mathbf{A}^\dagger$ satisfies the conditions

$$\mathbf{A}\mathbf{A}^T \text{ and } \mathbf{A}^T\mathbf{A} \text{ are symmetric, } \mathbf{A}\mathbf{A}^\dagger\mathbf{A} = \mathbf{A} \text{ and } \mathbf{A}^\dagger\mathbf{A}\mathbf{A}^\dagger = \mathbf{A}^\dagger$$

- **49(a)** $\mathbf{A} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \\ 1 & 1 \end{bmatrix}$ is of full rank 2 so pseudo inverse is

$$\mathbf{A}^\dagger = (\mathbf{A}^T\mathbf{A})^{-1}\mathbf{A}^T = \begin{bmatrix} 0.6364 & -0.3636 & 0.0909 \\ -0.3636 & 0.6364 & 0.0909 \end{bmatrix}$$

Equations (i) are consistent with unique solution

$$\begin{bmatrix} x \\ y \end{bmatrix} = \mathbf{A}^{\dagger} \begin{bmatrix} 3 \\ 3 \\ 2 \end{bmatrix} \Rightarrow x = y = 1$$

Equations (ii) are inconsistent with least squares solution

$$\begin{bmatrix} x \\ y \end{bmatrix} = \mathbf{A}^{\dagger} \begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix} \Rightarrow x = 1.0909, y = 1.0909$$

(b) $\mathbf{A} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \\ 10 & 10 \end{bmatrix}$ with pseudo inverse $\mathbf{A}^{\dagger} = \begin{bmatrix} 0.5072 & -0.4928 & 0.0478 \\ -0.4928 & 0.5072 & 0.0478 \end{bmatrix}$
Equations (i) are consistent with unique solution

$$\begin{bmatrix} x \\ y \end{bmatrix} = \mathbf{A}^{\dagger} \begin{bmatrix} 3 \\ 3 \\ 20 \end{bmatrix} \Rightarrow x = y = 1$$

Equations (ii) are inconsistent and have least squares solution

$$\begin{bmatrix} x \\ y \end{bmatrix} = \mathbf{A}^{\dagger} \begin{bmatrix} 3 \\ 3 \\ 30 \end{bmatrix} \Rightarrow x = y = 1.4785$$

(c) $\mathbf{A} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \\ 100 & 100 \end{bmatrix}$ with pseudo inverse $\mathbf{A}^{\dagger} = \begin{bmatrix} 0.5001 & -0.4999 & 0.0050 \\ -o.4999 & 0.5001 & 0.0050 \end{bmatrix}$
Equations (i) are consistent with unique solution

$$\begin{bmatrix} x \\ y \end{bmatrix} = \mathbf{A}^{\dagger} \begin{bmatrix} 3 \\ 3 \\ 200 \end{bmatrix} \Rightarrow x = y = 1$$

Equations (ii) are inconsistent with least squares solution

$$\begin{bmatrix} x \\ y \end{bmatrix} = \mathbf{A} \begin{bmatrix} 3 \\ 3 \\ 300 \end{bmatrix} \Rightarrow x = y = 1.4998$$

Since the sets of equations (i) are consistent weighting the last equation has no effect on the least squares solution which is unique. However, since the sets of equations (ii) are inconsistent the solution given is not unique but is the best in the least squares sense. Clearly as the weighting of the third equation increases from (a) to (b) to (c) the better is the matching to the third equation, and the last case (c) does not bother too much with the first two equations.

- **50** Data may be represented in the matrix form

$$\begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 2 & 1 \\ 3 & 1 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} m \\ c \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 2 \\ 2 \\ 3 \end{bmatrix}$$

$Az = Y$

MATLAB gives the pseudo inverse

$$\mathbf{A}^{\dagger} = \begin{bmatrix} -0.2 & -0.1 & 0 & 0.1 & 0.2 \\ 0.8 & 0.4 & 0.2 & 0 & -0.2 \end{bmatrix}$$

and, the least squares solution

$$\begin{bmatrix} m \\ c \end{bmatrix} = \mathbf{A}^{\dagger}\mathbf{y} = \begin{bmatrix} 0.5 \\ 0.8 \end{bmatrix}$$

leads to the linear model

$$y = 0.5x + 0.8$$

Exercises 1.9.3

- **51(a)** Taking $x_1 = y$

$$\dot{x}_1 = x_2 = \frac{dy}{dt}$$

$$\dot{x}_2 = x_3 = \frac{d^2y}{dt^2}$$

$$\dot{x}_3 = \frac{d^3y}{dt^3} = u(t) - 4x_1 - 5x_2 - 4x_3$$

Thus, state space form is

$$\dot{\mathbf{x}} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -4 & -5 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u(t)$$

$$y = x_1 = [1\ 0\ 0]\ [x_1\ x_2\ x_3]^T$$

51(b)

$$x_1 = y$$

$$x_2 = \dot{x}_1 = \frac{dy}{dt}$$

$$x_3 = \dot{x}_2 = \frac{d^2y}{dt^2}$$

$$x_4 = \dot{x}_3 = \frac{d^3y}{dt^3}$$

$$\dot{x}_4 = \frac{d^4y}{dt^4} = -4x_2 - 2x_3 + 5u(t)$$

Thus, state space form is

$$\dot{\mathbf{x}} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -4 & -2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 5 \end{bmatrix} u(t)$$

$$y = x_1 = [1\ 0\ 0\ 0]\ [x_1\ x_2\ x_3\ x_4]^T$$

■ **52(a)** Taking $\boldsymbol{A}$ to be the companion matrix of the LHS

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -7 & -5 & -6 \end{bmatrix}$$

and taking $\mathbf{b} = [0 \quad 0 \quad 1]^T$ and then using (1.67) in the text $\mathbf{c} = [5 \quad 3 \quad 1]$. Then from (1.84) the state-space form of the dynamic model is

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{b}u, \mathrm{y} = \mathbf{c}\mathbf{x}$$

(b) Taking $\boldsymbol{A}$ to be the companion matrix of the LHS

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -3 & -4 \end{bmatrix}$$

and taking $\mathbf{b} = [0 \quad 0 \quad 1]^T$ then using (1.67) in the text $\mathbf{c} = [2 \quad 3 \quad 1]$. Then from (1.84) the state-space form of the dynamic model is

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{b}u, \mathrm{y} = \mathbf{c}\mathbf{x}$$

■ **53** Applying Kirchhoff's second law to the individual loops gives

$$e = R_1(i_1 + i_2) + v_c + L_1\frac{di_1}{dt} \ , \ \dot{v}_c = \frac{1}{C}(i_1 + i_2)$$
$$e = R_1(i_1 + i_2) + v_c + L_2\frac{di_2}{dt} + R_2 i_2$$

so that,

$$\frac{di_1}{dt} = -\frac{R_1}{L_1}i_1 - \frac{R_1}{L_1}i_2 - \frac{v_c}{L_1} + \frac{e}{L_1}$$
$$\frac{di_2}{dt} = -\frac{R_1}{L_2}i_1 - \frac{(R_1 + R_2)}{L_2}i_2 - \frac{v_c}{L_2} + \frac{e}{L_2}$$
$$\frac{dv_c}{dt} = \frac{1}{C}(i_1 + i_2)$$

Taking $x_1 = i_1, x_2 = i_2, x_3 = v_c, u = e(t)$ gives the state equation as

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -\frac{R_1}{L_1} & -\frac{R_2}{L_1} & -\frac{1}{L_1} \\ -\frac{R_1}{L_2} & -\frac{(R_1+R_2)}{L_2} & -\frac{1}{L_2} \\ \frac{1}{C} & \frac{1}{C} & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} \frac{1}{L_1} \\ \frac{1}{L_2} \\ 0 \end{bmatrix} u(t) \qquad (1)$$

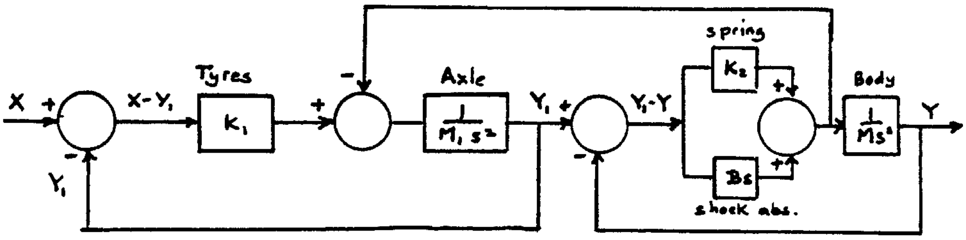
The output $y =$ voltage drop across $R_2 = R_2 i_2 = R_2 x_2$ so that

$$y = [0 \ R_2 \ 0] \ [x_1 \ x_2 \ x_3]^T \qquad (2)$$

Equations (1) and (2) are then in the required form

$$\dot{\mathbf{x}} = \mathbf{A} \ \mathbf{x} + \mathbf{b}u \ , \ y = \mathbf{c}^T\mathbf{x}$$

■ **54** The equations of motion, using Newton's second law, may be written down for the body mass and axle/wheel mass from which a state-space model can be deduced. Alternatively a block diagram for the system, which is more informative for modelling purposes, may be drawn up as follows



where s denotes the Laplace 's' and upper case variables X, Y, Y_1 denote the corresponding Laplace transforms of the corresponding lower case time domain variables $x(t), y(t), y_1(t); y_1(t)$ is the vertical displacement of the axle/wheel mass. Using basic block diagram rules this block diagram may be reduced to the input/output transfer function model

$$\xrightarrow{X} \boxed{\frac{K_1(K+Bs)}{(M_1s^2+K_1)(Ms^2+Bs+K)+Ms^2(K+Bs)}} \xrightarrow{Y}$$

or the time domain differential equation model

$$M_1M\frac{d^4y}{dt^4}+B(M_1+M)\frac{d^3y}{dt^3}+(K_1M+KM_1+KM)\frac{d^2y}{dt^2} \\ +K_1B\frac{dy}{dt}+K_1Ky=K_1K_2x+K_1B\frac{dx}{dt}$$

A possible state space model is

$$\begin{bmatrix}\dot{z}_1\\ \dot{z}_2\\ \dot{z}_3\\ \dot{z}_4\end{bmatrix}=\begin{bmatrix}-B(M_1+M) & 1 & 0 & 0\\ \frac{-(K_1M+KM_1+KM)}{MM_1} & 0 & 1 & 0\\ \frac{-K_1B}{M_1M} & 0 & 0 & 1\\ \frac{-K_1K}{M_1M} & 0 & 0 & 0\end{bmatrix}\begin{bmatrix}z_1\\ z_2\\ z_3\\ z_4\end{bmatrix}+\begin{bmatrix}0\\ 0\\ \frac{K_1B}{M_1M}\\ \frac{K_1K_2}{MM_1}\end{bmatrix}x(t)$$

$$y=[1\ 0\ 0\ 0]\mathbf{z}(t),\ \mathbf{z}=[z_1\ z_2\ z_3\ z_4]^T.$$

Clearly alternative forms may be written down, such as, for example, the companion form of equation (1.66) in the text. Disadvantage is that its output y is not one of the state variables.

■ **55** Applying Kirchhoff's second law to the first loop gives

$$x_1 + R_3(i - i_1) + R_1 i = u$$
$$\text{that is, } (R_1 + R_3)i - R_3 i_1 + x_1 = u$$

Applying it to the outer loop gives

$$x_2 + (R_4 + R_2)i_1 + R_1 i = u$$

Taking $\alpha = R_1R_3 + (R_1 + R_3)(R_4 + R_2)$ then gives

$$\alpha i = (R_2 + R_3 + R_4)u - (R_2 + R_4)x_1 - R_3x_2$$
$$\text{and} \quad \alpha i_1 = R_3u + R_1x_1 - (R_1 + R_3)x_2$$

Thus,

$$\alpha(i - i_1) = (R_4 + R_2)u - (R_1 + R_2 + R_4)x_1 + R_1x_2$$

$$\text{Voltage drop across } C_1 : \dot{x}_1 = \frac{1}{C_1}(i - i_1)$$
$$= \frac{1}{\alpha C_1}[-(R_1 + R_2 + R_4)x_1 + R_1x_2 + (R_4 + R_2)u] \quad (1)$$

$$\text{Voltage drop across } C_2 : \dot{x}_2 = \frac{1}{C_2}i_1$$
$$= \frac{1}{\alpha C_2}[R_1x_1 - (R_1 + R_3)x_2 + R_3u] \quad (2)$$

$$y_1 = i_1 = \frac{R_1}{\alpha}x - \frac{(R_1 + R_3)}{\alpha}x_2 + \frac{R_3}{\alpha}u \quad (3)$$
$$y_2 = R_2(i - i_1) = -\frac{R_3}{\alpha}(R_1 + R_2 + R_4)x_1 + \frac{R_3R_1}{\alpha}x_2 + R_3\frac{(R_4 + R_2)}{\alpha}u \quad (4)$$

Equations (1)–(4) give the required state space model.
Substituting the given values for R_1, R_2, R_3, R_4, C_1 and C_2 gives the state matrix $\mathbf{A}$ as

$$\mathbf{A} = \begin{bmatrix} \frac{-9}{35.10^{-3}} & \frac{1}{35.10^{-3}} \\ \frac{1}{35.10^{-3}} & \frac{-4}{35.10^{-3}} \end{bmatrix} = \frac{10^3}{35}\begin{bmatrix} -9 & 1 \\ 1 & -4 \end{bmatrix}$$

Let $\beta = \dfrac{10^3}{35}$ then eigenvalues are solutions of

$$\begin{vmatrix} -9\beta - \lambda & \beta \\ \beta & -4\beta - \lambda \end{vmatrix} = \lambda^2 + 13\beta\lambda + 35\beta^2 = 0$$

giving

$$\lambda = \frac{-13 \pm \sqrt{29}}{2}\beta \simeq -2.6 \times 10^2 \text{ or } -1.1 \times 10^2$$

Exercises 1.10.4

■ **56** $\quad \mathbf{\Phi}(t) = e^{\mathbf{A}t}$ where $\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$

Eigenvalues of $\mathbf{A}$ are $\lambda = 1, \lambda = 1$ so

$$e^{\mathbf{A}t} = \alpha_0(t)\mathbf{I} + \alpha_1(t)\mathbf{A}$$

where α_0, α_1 satisfy

$$e^{\lambda t} = \alpha_0 + \alpha_1\lambda, \qquad \lambda = 1$$
$$te^{\lambda t} = \alpha_1$$

giving $\alpha_1 = te^t$, $\alpha_0 = e^t - te^t$

Thus,

$$\mathbf{\Phi}(t) = e^{\mathbf{A}t} = \begin{bmatrix} e^t - te^t & 0 \\ 0 & e^t - te^t \end{bmatrix} + \begin{bmatrix} te^t & 0 \\ te^t & te^t \end{bmatrix} = \begin{bmatrix} e^t & 0 \\ te^t & e^t \end{bmatrix}$$

56(a) $\quad \mathbf{\Phi}(0) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \mathbf{I}$

56(b)

$$\mathbf{\Phi}(t_2 - t_1)\mathbf{\Phi}(t_1) = \begin{bmatrix} e^{t_2}e^{-t_1} & 0 \\ (t_2 - t_1)e^{t_2}e^{-t_1} & e^{t_2}e^{-t_1} \end{bmatrix} \begin{bmatrix} e^{t_1} & 0 \\ t_1e^{t_1} & e^{t_1} \end{bmatrix}$$
$$= \begin{bmatrix} e^{t_2} & 0 \\ (t_2 - t_1)e^{t_2} + t_1e^{t_2} & e^{t_2} \end{bmatrix} = \begin{bmatrix} e^{t_2} & 0 \\ t_2e^{t_2} & e^{t_2} \end{bmatrix} = \mathbf{\Phi}(t_2)$$

56(c) $\quad \mathbf{\Phi}^{-1} = \dfrac{1}{e^{2t}} \begin{bmatrix} e^t & 0 \\ -te^t & e^t \end{bmatrix} = \begin{bmatrix} e^{-t} & 0 \\ -te^{-t} & e^{-t} \end{bmatrix} = \mathbf{\Phi}(-t)$

■ **57** Take $x_1 = y,\ x_2 = \dot{x}_1 = \dfrac{dy}{dt},\ \dot{x}_2 = \dfrac{d^2y}{dt^2} = -x_1 - 2x_2$ so in vector–matrix form the differential equation is

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix} \mathbf{x},\ y = [1\ 0]\mathbf{A}$$

Taking $\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix}$ its eigenvalues are $\lambda = -1, \lambda = -1$

$$\begin{aligned} e^{\mathbf{A}t} &= \alpha_0 \mathbf{I} + \alpha_1 \mathbf{A} \text{ where } \alpha_0, \alpha_1 \text{ satisfy} \\ e^{\lambda t} &= \alpha_0 + \alpha_1 \lambda,\ \ \lambda = -1 \\ te^{\lambda t} &= \alpha_1 \end{aligned}$$

giving $\alpha_0 = e^{-t} + te^{-t}, \alpha_1 = te^{-t}$. Thus,

$$e^{\mathbf{A}t} = \begin{bmatrix} e^{-t} + te^{-t} & te^{-t} \\ -te^{-t} & e^{-t} - te^{-t} \end{bmatrix}$$

Thus, solution of differential equation is

$$\begin{aligned} \mathbf{x}(t) &= e^{\mathbf{A}t}\mathbf{x}(0),\ \mathbf{x}(0) = [1\ 1]^T \\ &= \begin{bmatrix} e^{-t} + 2te^{-t} \\ e^{-t} - 2te^{-t} \end{bmatrix} \end{aligned}$$

giving $y(t) = x_1(t) = e^{-t} + 2te^{-t}$

The differential equation may be solved directly using the techniques of Chapter 10 of the companion text *Modern Engineering Mathematics* or using Laplace transforms. Both approaches confirm the solution

$$y = (1 + 2t)e^{-2t}$$

■ **58** Taking $\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ then from Exercise 56

$$e^{\mathbf{A}t} = \begin{bmatrix} e^t & 0 \\ te^t & e^t \end{bmatrix}$$

and the required solution is

$$\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{x}(0) = \begin{bmatrix} e^t & 0 \\ te^t & e^t \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} e^t \\ (1+t)e^t \end{bmatrix}$$

■ **59** Taking $\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -6 & -5 \end{bmatrix}$ its eigenvalues are $\lambda_1 = -3,\ \lambda_2 = -2$.
Thus, $e^{\mathbf{A}t} = \alpha_0\mathbf{I} + \alpha_1\mathbf{A}$ where α_0, α_1 satisfy

$$e^{-3t} = \alpha_0 - 3\alpha_1, \quad e^{-2t} = \alpha_0 - 2\alpha_1$$
$$\alpha_0 = 3e^{-2t} - 2e^{-3t},\ \alpha_1 = e^{-2t} - e^{-3t}$$

so

$$e^{\mathbf{A}t} = \begin{bmatrix} 3e^{-2t} - 2e^{-3t} & e^{-2t} - e^{-3t} \\ 6e^{-3t} - te^{-2t} & 3e^{-3t} - 2e^{-2t} \end{bmatrix}$$

Thus, the first term in (6.73) becomes

$$e^{\mathbf{A}t}\mathbf{x}(0) = e^{\mathbf{A}t}[1 \ \ -1]^T = \begin{bmatrix} 2e^{-2t} - e^{-3t} \\ 3e^{-3t} - 4e^{-2t} \end{bmatrix}$$

and the second term is

$$\begin{aligned} \int_0^t e^{\mathbf{A}(t-\tau)}\mathbf{b}u(\tau)d\tau &= \int_0^t 2 \begin{bmatrix} 6e^{-2(t-\tau)} - 6e^{-3(t-\tau)} \\ 18e^{-3(t-\tau)} - 12e^{-2(t-\tau)} \end{bmatrix} d\tau \\ &= 2 \begin{bmatrix} 3e^{-2(t-\tau)} - 2e^{-3(t-\tau)} \\ 6e^{-3(t-\tau)} - 6e^{-2(t-\tau)} \end{bmatrix}_0^t \\ &= 2 \begin{bmatrix} 1 - 3e^{-2t} + 2e^{-3t} \\ 6e^{-2t} - 6e^{-3t} \end{bmatrix} \end{aligned}$$

Thus, required solution is

$$\begin{aligned} \mathbf{x}(t) &= \begin{bmatrix} 2e^{-2t} - e^{-3t} + 2 - 6e^{-3t} + 4e^{-3t} \\ 3e^{-3t} - 4e^{-2t} + 12e^{-2t} - 12e^{-3t} \end{bmatrix} \\ &= \begin{bmatrix} 2 - 4e^{-2t} + 3e^{-3t} \\ 8e^{-2t} - 9e^{-3t} \end{bmatrix} \end{aligned}$$

that is, $x_1 = 2 - 4e^{-2t} + 3e^{-3t},\ x_2 = 8e^{-2t} - 9e^{-3t}$

■ **60** In state space form,

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 2 \\ 0 \end{bmatrix} u(t), \; u(t) = e^{-t}, \; x(0) = [0 \; 1]^T$$

Taking $\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$ its eigenvalues are $\lambda_1 = -2, \lambda_2 = -1$ so

$$e^{\mathbf{A}t} = \alpha_0 I + \alpha_1 \mathbf{A} \text{ where } \alpha_0, \alpha_1 \text{ satisfy}$$

$$e^{-2t} = \alpha_0 - 2\alpha_1, \; e^{-t} = \alpha_0 - \alpha_1 \;\Rightarrow\; \alpha_0 = 2e^{-t} - e^{-2t}, \; \alpha_1 = e^{-t} - e^{-2t}$$

Thus,

$$e^{\mathbf{A}t} = \begin{bmatrix} 2e^{-t} - e^{-2t} & e^{-t} - e^{-2t} \\ -2e^{-t} + 2e^{-2t} & -e^{-t} + 2e^{-2t} \end{bmatrix}$$

and $e^{\mathbf{A}t}\mathbf{x}(0) = \begin{bmatrix} e^{-t} - e^{-2t} \\ -e^{-t} + 2e^{-2t} \end{bmatrix}$

$$\begin{aligned} \int_0^t \mathbf{A}^{(t-\tau)} \mathbf{b} u(\tau) d\tau &= \int_0^t \begin{bmatrix} 4e^{-(t-\tau)} - 2e^{-2(t-\tau)} \\ -4e^{-(t-\tau)} + 4e^{-2(t-\tau)} \end{bmatrix} e^{-\tau} d\tau \\ &= \int_0^t \begin{bmatrix} 4e^{-t} - 2e^{-2t}e^{\tau} \\ -4e^{-t} + 4e^{-2t}e^{\tau} \end{bmatrix} d\tau \\ &= \begin{bmatrix} 4\tau e^{-t} - 2e^{-2t}e^{\tau} \\ -4\tau e^{-t} + 4e^{-2t}e^{\tau} \end{bmatrix}_0^t \\ &= \begin{bmatrix} 4te^{-t} - 2e^{-t} + 2e^{-2t} \\ -4te^{-t} + 4e^{-t} - 4e^{-2t} \end{bmatrix} \end{aligned}$$

We therefore have the solution

$$\begin{aligned} \mathbf{x}(t) &= e^{\mathbf{A}t}\mathbf{x}(0) + \int_0^t e^{A(t-\tau)} \mathbf{b} u(\tau) d\tau \\ &= \begin{bmatrix} 4te^{-t} + e^{-2t} - e^{-t} \\ -4te^{-t} + 3e^{-t} - 2e^{-2t} \end{bmatrix} \end{aligned}$$

that is,

$$x_1 = 4te^{-t} + e^{-2t} - e^{-t}, \; x_2 = -4te^{-t} + 3e^{-t} - 2e^{-2t}$$

■ **61** Taking $\mathbf{A} = \begin{bmatrix} 3 & 4 \\ 2 & 1 \end{bmatrix}$ its eigenvalues are $\lambda_1 = 5, \lambda_2 = -1$.

$$e^{\mathbf{A}t} = \alpha_0 \mathbf{I} + \alpha_1 \mathbf{A} \text{ where } \alpha_0, \alpha_1 \text{ satisfy}$$

$$e^{5t} = \alpha_0 + 5\alpha_1, \; e^{-t} = \alpha_0 - \alpha_1 \;\Rightarrow\; \alpha_0 = \frac{1}{6}e^{5t} + \frac{5}{6}e^{-t}, \; \alpha_1 = \frac{1}{6}e^{5t} + \frac{1}{6}e^{-t}$$

Thus, transition matrix is

$$e^{\mathbf{A}t} = \begin{bmatrix} \frac{1}{3}e^{-t} + \frac{2}{3}e^{5t} & \frac{2}{3}e^{5t} - \frac{2}{3}e^{-t} \\ \frac{1}{3}e^{5t} - \frac{1}{3}e^{-t} & \frac{1}{3}e^{5t} + \frac{2}{3}e^{-t} \end{bmatrix}$$

and $e^{\mathbf{A}t}\mathbf{x}(0) = \mathbf{e}^{\mathbf{A}t}[1\ 2]^T = \begin{bmatrix} 2e^{5t} - e^{-t} \\ e^{5t} + e^{-t} \end{bmatrix}$

$$\begin{aligned}
\int_0^t e^{\mathbf{A}(t-\tau)}\mathbf{B}u(\tau)d\tau &= \int_0^t e^{\mathbf{A}(t-\tau)} \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ 3 \end{bmatrix} d\tau \\
&= \int_0^t \mathbf{A}^{t-\tau} \begin{bmatrix} 3 \\ 7 \end{bmatrix} d\tau \\
&= \int_0^t \begin{bmatrix} \frac{20}{3}e^{5(t-\tau)} - \frac{11}{3}e^{-(t-\tau)} \\ \frac{10}{3}e^{5(t-\tau)} + \frac{11}{3}e^{-(t-\tau)} \end{bmatrix} d\tau \\
&= \begin{bmatrix} -\frac{4}{3}e^{5(t-\tau)} - \frac{11}{3}e^{-(t-\tau)} \\ -\frac{2}{3}e^{5(t-\tau)} + \frac{11}{3}e^{-(t-\tau)} \end{bmatrix}_0^t \\
&= \begin{bmatrix} -5 + \frac{11}{3}e^{-t} + \frac{4}{3}e^{5t} \\ 3 - \frac{11}{3}e^{-t} + \frac{2}{3}e^{5t} \end{bmatrix}
\end{aligned}$$

Thus, solution is

$$\begin{aligned}
\mathbf{x}(t) &= e^{\mathbf{A}t}\mathbf{x}(0) + \int_0^t e^{\mathbf{A}(t-\tau)}\mathbf{B}u(t)d\tau \\
&= \begin{bmatrix} -5 + \frac{8}{3}e^{-t} + \frac{10}{3}e^{5t} \\ 3 - \frac{8}{3}e^{-t} + \frac{5}{3}e^{5t} \end{bmatrix}
\end{aligned}$$

Exercises 1.10.7

■ **62** Eigenvalues of matrix $\mathbf{A} = \begin{bmatrix} -\frac{3}{2} & \frac{3}{4} \\ 1 & -\frac{5}{2} \end{bmatrix}$ are given by

$$| A - \lambda I | = \lambda^2 + 4\lambda + 3 = (\lambda + 3)(\lambda + 1) = 0$$

that is, $\lambda_1 = -1,\ \lambda_2 = -3$
having corresponding eigenvectors $\mathbf{e}_1 = [3\ 2]^T,\ \mathbf{e}_2 = [1\ \ -2]^T$.
Denoting the reciprocal basis vectors by

$$\mathbf{r}_1 = [r_{11}\ r_{12}]^T,\ \mathbf{r}_2 = [r_{21}\ r_{22}]^T$$

and using the relationships $\mathbf{r}_i^T\mathbf{e}_j = \delta_{ij}(i, j = 1, 2)$ we have

$$\left.\begin{aligned} 3r_{11} + 2r_{12} &= 1 \\ r_{11} - 2r_{12} &= 0 \end{aligned}\right\} \quad \mathbf{r}_1 = [\tfrac{1}{4}\ \tfrac{1}{8}]^T$$

$$\left.\begin{aligned} 3r_{21} + 2r_{22} &= 0 \\ r_{21} - 2r_{22} &= 1 \end{aligned}\right\} \quad \mathbf{r}_2 = [\tfrac{1}{4}\ -\tfrac{3}{8}]^T$$

Thus,

$$\mathbf{r}_1^T\mathbf{x}(0) = \frac{1}{2} + \frac{1}{2} = 1,\ \mathbf{r}_2^T\mathbf{x}(0) = \frac{1}{2} - \frac{3}{2} = -1$$

so the spectral form of solution is

$$\mathbf{x}(t) = e^{-t}\mathbf{e}_1 - e^{-3t}\mathbf{e}_2$$

The trajectory is readily drawn showing that it approaches the origin along the eigenvector $\mathbf{e}_1$ since $e^{-3t} \to 0$ faster than e^{-t}. See Figure 1.9 in the text.

■ **63** Taking $\mathbf{A} = \begin{bmatrix} -2 & 2 \\ 2 & -5 \end{bmatrix}$ eigenvalues are $\lambda_1 = -6,\ \lambda_2 = -1$ having corresponding eigenvectors $\mathbf{e}_1 = [1\ -2]^T,\ \mathbf{e}_2 = [2\ 1]^T$.
Denoting the reciprocal basis vectors by

$$\mathbf{r}_1 = [r_{11}\ r_{12}]^T,\ \mathbf{r}_2 = [r_{21}\ r_{22}]^T$$

and using the relationships $\mathbf{r}_i^T\mathbf{e}_j = \delta_{ij}(i, j = 1, 2)$ we have

$$\left.\begin{aligned} r_{11} - 2r_{12} &= 1 \\ 2r_{11} + r_{12} &= 0 \end{aligned}\right\} \quad \Rightarrow\ r_{11} = \tfrac{1}{5}, r_{12} = -\tfrac{2}{5}\ \Rightarrow\ \mathbf{r}_1 = \tfrac{1}{5}[1\ -2]^T$$

$$\left.\begin{aligned} r_{21} - 2r_{22} &= 0 \\ 2r_{21} + r_{22} &= 1 \end{aligned}\right\} \quad \Rightarrow\ r_{21} = \tfrac{2}{5}, r_{22} = -\tfrac{1}{5}\ \Rightarrow\ \mathbf{r}_2 = \tfrac{1}{5}[2\ 1]^T$$

Thus,

$$\mathbf{r}_1^T\mathbf{x}(0) = \frac{1}{5}[1 - 2]\begin{bmatrix} 2 \\ 3 \end{bmatrix} = -\frac{4}{5}$$

$$\mathbf{r}_2^T\mathbf{x}(0) = \frac{1}{5}[2\ 1]\begin{bmatrix} 2 \\ 3 \end{bmatrix} = \frac{7}{5}$$

then response is

$$\mathbf{x}(t) = \sum_{i=1}^{2} \mathbf{r}_i^T \mathbf{x}(0) e^{\lambda it} \mathbf{e}_i$$
$$= -\frac{4}{5} e^{-6t} \begin{bmatrix} 1 \\ -2 \end{bmatrix} + \frac{7}{5} e^{-t} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \frac{1}{5} \begin{bmatrix} -4e^{-6t} + 14e^{-t} \\ 8e^{-6t} + 7e^{-t} \end{bmatrix}$$

Again, following Figure 1.9 in the text, the trajectory is readily drawn and showing that it approaches the origin along the eigenvector $\mathbf{e}_2$ since $e^{-6t} \to 0$ faster than e^{-t}.

■ **64** Taking $\mathbf{A} = \begin{bmatrix} 0 & -4 \\ 2 & -4 \end{bmatrix}$ eigenvalues are $\lambda_1 = -2 + j2,\ \lambda_2 = -2 - j2$ having corresponding eigenvectors $\mathbf{e}_1 = [2 \ \ 1 - j]^T, \mathbf{e}_2 = [2 \ \ 1 + j]^T$.

Let $\mathbf{r}_1 = \mathbf{r}_1' + j\mathbf{r}_1''$ be reciprocal base vector to $\mathbf{e}_1$ then

$$\mathbf{r}_1^T \mathbf{e}_1 = 1 = [r' + j\mathbf{r}_1'']^T [\mathbf{e}_1' + je_1'']^T \text{ where } \mathbf{e}_1 = \mathbf{e}_1' + j\mathbf{e}_1''$$
$$\mathbf{r}_1^T \mathbf{e}_2 = 0 = [\mathbf{r}_1' + j\mathbf{r}_1'']^T [\mathbf{e}_1' - j\mathbf{e}_1'']^T \text{ since } \mathbf{e}_2 = \text{conjugate } \ \mathbf{e}_1$$

Thus,

$$[(\mathbf{r}_1')^T \mathbf{e}_1' - (r_1'')^T \mathbf{e}_1''] + j[(r_1'')^T \mathbf{e}_1' + (r_1')^T \mathbf{e}_1''] = 1$$

and

$$[(\mathbf{r}_1')^T \mathbf{e}_1' - (r_1'')^T \mathbf{e}_1'] + j[(r_1')^T \mathbf{e}_1^1 - (r_1')^T \mathbf{e}_1'] = 0$$

giving

$$(\mathbf{r}_1')^T\mathbf{e}_1' = \frac{1}{2}\ ,\ (\mathbf{r}_1')^T\mathbf{e}_1' = \frac{1}{2}\ ,\ (\mathbf{r}_1')^T\mathbf{e}_1' = (\mathbf{r}_1')^T\mathbf{e}_1'' = 0$$

Now $\mathbf{e}_1' = [2\ 1]^T, \mathbf{e}_1'' = [0\ \ -1]^T$

Let $\mathbf{r}_1' = [a\ b]^T$ and $\mathbf{r}_1'' = [c\ d]^T$ then from above

$2a + b = \frac{1}{2}, -b = 0$ and $-d = -\frac{1}{2}, 2c + d = 0$

giving $a = \frac{1}{4}, b = 0, c = -\frac{1}{4}, d = \frac{1}{2}$ so that

$$\mathbf{r}_1 = \mathbf{r}_1' + j\mathbf{r}_1'' = \frac{1}{4}[1 - j\ \ 2j]^T$$

Since $\mathbf{r}_2$ is the complex conjugate of $\mathbf{r}_1$

$$\mathbf{r}_2 = \frac{1}{4}[1 + j\ \ -2j]^T$$

so the solution is given by

$$\mathbf{x}(t) = \mathbf{r}_1^T\mathbf{x}(0)e^{\lambda_1 t}\mathbf{e}_1 + \mathbf{r}_2^T\mathbf{x}(0)e^{\lambda_2 t}\mathbf{e}_2$$

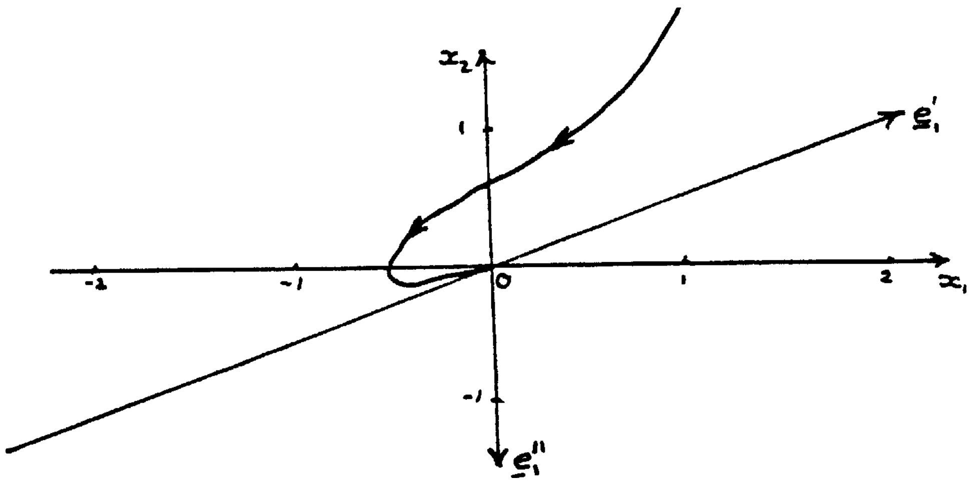
and since $\mathbf{r}_1^T\mathbf{x}(0) = \frac{1}{2}(1 + j),\ \mathbf{r}_2^T\mathbf{x}(0) = \frac{1}{2}(1 - j)$

$$x(t) = e^{-2t}\left\{\frac{1}{2}(1 + j)e^{2jt}\ \begin{bmatrix} 2 \\ 1 - j \end{bmatrix}\ + \frac{1}{2}(1 - j)e^{-2jt}\ \begin{bmatrix} 2 \\ 1 + j \end{bmatrix}\right\}$$

$$= e^{-2t}\left\{(\cos 2t - \sin 2t)\ \begin{bmatrix} 2 \\ 1 \end{bmatrix} - (\cos 2t + \sin 2t)\ \begin{bmatrix} 0 \\ -1 \end{bmatrix}\right\}$$

$$= e^{-2t}\left\{(\cos 2t - \sin 2t)\mathbf{e}_1' - (\cos 2t + \sin 2t)\mathbf{e}_1''\right\} \text{ where } \mathbf{e}_1 = \mathbf{e}_1' + j\mathbf{e}_1''$$

To plot the trajectory, first plot $\mathbf{e}_1', \mathbf{e}_1''$ in the plane and then using these as a frame of reference plot the trajectory. A sketch is as follows



■ **65** Following section 1.10.6 if the equations are representative of

$$\dot{\mathbf{x}} = \mathbf{A}\ \mathbf{x} + \mathbf{b}u\ ,\ y = \mathbf{c}^T\mathbf{x}$$

then making the substitution $\mathbf{x} = \mathbf{M}\ \boldsymbol{\xi}$, where $\mathbf{M}$ is the modal matrix of $\mathbf{A}$, reduces the system to the canonical form

$$\dot{\boldsymbol{\xi}} = \boldsymbol{\Lambda}\ \boldsymbol{\xi} + (\mathbf{M}^{-1}\mathbf{b})u\ ,\ y = (\mathbf{c}^T\mathbf{M})\boldsymbol{\xi}$$

where $\boldsymbol{\Lambda}$ is the spectral matrix of $\mathbf{A}$.
Eigenvalues of $\mathbf{A}$ are given by

$$\begin{vmatrix} 1-\lambda & 1 & -2 \\ -1 & 2-\lambda & 1 \\ 0 & 1 & -1-\lambda \end{vmatrix} = \lambda^3 - 2\lambda^2 - \lambda + 2 = (\lambda - 1)(\lambda + 2)(\lambda + 1) = 0$$

so the eigenvalues are $\lambda_1 = 2, \lambda_2 = 1, \lambda_3 = -1$. The corresponding eigenvectors are readily determined as

$$\mathbf{e}_1 = [1\ 3\ 1]^T,\ \mathbf{e}_2 = [3\ 2\ 1]^T,\ \mathbf{e}_3 = [1\ 0\ 1]^T$$

Thus, $\mathbf{M} = \begin{bmatrix} 1 & 3 & 1 \\ 3 & 2 & 0 \\ 1 & 1 & 1 \end{bmatrix}$ and $\boldsymbol{\Lambda} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$

$\mathbf{M}^{-1} = \dfrac{1}{\det \mathbf{M}}$ adj $\mathbf{M} = -\dfrac{1}{6}\begin{bmatrix} 2 & -2 & -2 \\ -3 & 0 & 3 \\ 1 & 2 & -7 \end{bmatrix}$ so required canonical form is

$$\begin{bmatrix} \dot{\xi}_1 \\ \dot{\xi}_2 \\ \dot{\xi}_3 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{bmatrix} + \begin{bmatrix} \frac{1}{3} \\ 0 \\ -\frac{4}{3} \end{bmatrix} u$$

$$y = [1 \; -4 \; -2] \; [\xi_1 \; \xi_2 \; \xi_3]^T$$

■ **66** Let $\mathbf{r}_1 = [r_{11} \; r_{12} \; r_{13}]^T, \mathbf{r}_2 = [r_{21} \; r_{22} \; r_{23}]^T, \mathbf{r}_3 = [r_{31} \; r_{32} \; r_{33}]^T$ be the reciprocal base vectors to $\mathbf{e}_1 = [1 \; 1 \; 0]^T, \; \mathbf{e}_2 = [0 \; 1 \; 1]^T, \; \mathbf{e}_3 = [1 \; 2 \; 3]^T$.

$$\left.\begin{aligned} \mathbf{r}_1^T\mathbf{e}_1 &= r_{11} + r_{12} = 1 \\ \mathbf{r}_1^T\mathbf{e}_2 &= r_{11} + r_{13} = 0 \\ \mathbf{r}_1^T\mathbf{e}_3 &= r_{11} + 2r_{12} + 3r_{13} = 0 \end{aligned}\right\} \Rightarrow \mathbf{r}_1 = \frac{1}{2}[1 \; 1 \; -1]^T$$

$$\left.\begin{aligned} \mathbf{r}_2^T\mathbf{e}_1 &= r_{21} + r_{22} = 0 \\ \mathbf{r}_2^T\mathbf{e}_2 &= r_{22} + r_{23} = 1 \\ \mathbf{r}_2^T\mathbf{e}_3 &= r_{21} + 2r_{22} + 3r_{23} = 0 \end{aligned}\right\} \Rightarrow \mathbf{r}_2 = \frac{1}{2}[-3 \; 3 \; 1]^T$$

$$\left.\begin{aligned} \mathbf{r}_3^T\mathbf{e}_1 &= r_{31} + r_{32} = 0 \\ \mathbf{r}_3^T\mathbf{e}_2 &= r_{32} + r_{33} = 0 \\ \mathbf{r}_3^T\mathbf{e}_3 &= r_{31} + 2r_{32} + 3r_{33} = 1 \end{aligned}\right\} \Rightarrow \mathbf{r}_3 = \frac{1}{2}[1 \; -1 \; 1]^T$$

Then using the fact that $\mathbf{x}(0) = [1 \; 1 \; 1]^T$

$$\alpha_0 = \mathbf{r}_1^T\mathbf{x}(0) = -\tfrac{1}{2} \;, \; \alpha_1 = \mathbf{r}_2^T\mathbf{x}(0) = \tfrac{1}{2} \;, \; \alpha_3 = \mathbf{r}_3^T\mathbf{x}(0) = \tfrac{1}{2}$$

■ **67** The eigenvectors of $\mathbf{A}$ are given by

$$\begin{vmatrix} 5-\lambda & 4 \\ 1 & 2-\lambda \end{vmatrix} = (\lambda - 6)(\lambda - 1) = 0$$

so the eigenvalues are $\lambda_1 = 6, \lambda_2 = 1$. The corresponding eigenvectors are readily determined as $\mathbf{e}_1 = [4 \; 1]^T, \mathbf{e}_2 = [1 \; -1]^T$.

Taking $\mathbf{M}$ to be the modal matrix $\mathbf{M} = \begin{bmatrix} 4 & 1 \\ 1 & -1 \end{bmatrix}$ then substituting $\mathbf{x} = M\boldsymbol{\xi}$ into $\dot{\mathbf{x}} = \mathbf{A}x(t)$ reduces it to the canonical form

$$\dot{\boldsymbol{\xi}} = \boldsymbol{\Lambda}\boldsymbol{\xi}$$

where $\mathbf{\Lambda} = \begin{bmatrix} 6 & 0 \\ 0 & 1 \end{bmatrix}$. Thus, the decoupled canonical form is

$$\begin{bmatrix} \dot{\xi}_1 \\ \dot{\xi}_2 \end{bmatrix} = \begin{bmatrix} 6 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} \text{ or } \dot{\xi}_1 = 6\xi_1 \text{ and } \dot{\xi}_2 = \xi_2$$

which may be individually solved to give

$$\xi_1 = \alpha e^{6t} \text{ and } \xi_1 = \beta e^t$$

Now $\boldsymbol{\xi}(0) = \mathbf{M}^{-1}\mathbf{x}(0) = -\frac{1}{5}\begin{bmatrix} -1 & -1 \\ -1 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 4 \end{bmatrix} = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$
so $\xi_1(0) = 1 = \alpha$ and $\xi_2(0) = -3 = \beta$
giving the solution of the uncoupled system as

$$\boldsymbol{\xi} = \begin{bmatrix} e^{6t} \\ -3e^t \end{bmatrix}$$

The solution for $\mathbf{x}(t)$ as

$$\mathbf{x} = \mathbf{M}\ \boldsymbol{\xi} = \begin{bmatrix} 4 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} e^{6t} \\ -3e^t \end{bmatrix} = \begin{bmatrix} 4e^{6t} - 3e^t \\ e^{6t} + 3e^t \end{bmatrix}$$

■ **68** Taking $\mathbf{A} = \begin{bmatrix} 3 & 4 \\ 2 & 1 \end{bmatrix}$ its eigenvalues are $\lambda_1 = 5, \lambda_2 = -1$ having corresponding eigenvectors $\mathbf{e}_1 = [2\ 1]^T$, $\mathbf{e}_2 = [1\ -1]^T$.
Let $\mathbf{M} = \begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix}$ be the modal matrix of $\mathbf{A}$, then $\dot{\mathbf{x}} = \mathbf{M}\ \boldsymbol{\xi}$ reduces the equation to

$$\dot{\boldsymbol{\xi}}(t) = \begin{bmatrix} 5 & 0 \\ 0 & -1 \end{bmatrix} \boldsymbol{\xi} + \mathbf{M}^{-1} \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \mathbf{u}(t)$$

Since $\mathbf{M}^{-1} = \frac{1}{\det \mathbf{M}}$ adj $\mathbf{M} = \frac{1}{3}\begin{bmatrix} 1 & 1 \\ 1 & -2 \end{bmatrix}$ we have,

$$\dot{\boldsymbol{\xi}}(t) = \begin{bmatrix} 5 & 0 \\ 0 & -1 \end{bmatrix} \boldsymbol{\xi} + \frac{1}{3}\begin{bmatrix} 1 & 2 \\ -2 & -1 \end{bmatrix} \mathbf{u}(t)$$

With $\mathbf{u}(t) = [4\ 3]^T$ the decoupled equations are

$$\dot{\xi}_1 = 5\xi_1 + \frac{10}{3}$$
$$\dot{\xi}_2 = -\xi_2 - \frac{11}{3}$$

which can be solved independently to give

$$\xi_1 = \alpha e^{5t} - \frac{2}{3}\ ,\ \xi_2 = \beta e^{-t} - \frac{11}{3}$$

We have that $\boldsymbol{\xi}(0) = \boldsymbol{M}^{-1}\mathbf{x}(0) = \frac{1}{3}\begin{bmatrix}1 & 1\\ 1 & -2\end{bmatrix}\begin{bmatrix}1\\ 2\end{bmatrix} = \begin{bmatrix}1\\ -1\end{bmatrix}$ so

$$\begin{aligned} 1 &= \alpha - \frac{2}{3} \ \Rightarrow\ \alpha = \frac{5}{3}\\ -1 &= \beta - \frac{11}{3} \ \Rightarrow\ \beta = \frac{8}{3} \end{aligned}$$

giving

$$\boldsymbol{\xi} = \begin{bmatrix}\frac{5}{3}e^{5t} - \frac{2}{3}\\ \frac{8}{3}e^{-t} - \frac{11}{3}\end{bmatrix}$$

and $\mathbf{x} = \boldsymbol{M}\boldsymbol{\xi} = \begin{bmatrix}2 & 1\\ 1 & -1\end{bmatrix}\begin{bmatrix}\frac{5}{3}e^{5t} - \frac{2}{3}\\ \frac{8}{3}e^{-t} - \frac{11}{3}\end{bmatrix} = \begin{bmatrix}-5 + \frac{8}{3}e^{-t} + \frac{10}{3}e^{5t}\\ 3 - \frac{8}{3}e^{-t} + \frac{5}{3}e^{5t}\end{bmatrix}$

which confirms Exercises 57 and 58.

Exercises 1.11.1 (Lyapunov)

▪ **69** Take tentative Lyapunov function $V(\mathbf{x}) = \mathbf{x}^T\mathbf{P}\mathbf{x}$ giving

$$\dot{V}(\mathbf{x}) = \mathbf{x}^T(\mathbf{A}^T\mathbf{P} + \mathbf{P}\mathbf{A})\mathbf{x} = -\mathbf{x}^T\mathbf{Q}\mathbf{x} \text{ where}$$

$$\mathbf{A}^T\mathbf{P} + \mathbf{P}\mathbf{A} = -\mathbf{Q} \qquad \text{(i)}$$

Take $\mathbf{Q} = \mathbf{I}$ so that $\dot{V}(\mathbf{x}) = -(x_1^2 + x_2^2)$ which is negative definite. Substituting in (i) gives

$$\begin{bmatrix}-4 & 3\\ 2 & -2\end{bmatrix}\begin{bmatrix}p_{11} & p_{12}\\ p_{12} & p_{22}\end{bmatrix} + \begin{bmatrix}p_{11} & p_{12}\\ p_{12} & p_{22}\end{bmatrix}\begin{bmatrix}-4 & 2\\ 3 & -2\end{bmatrix} = \begin{bmatrix}-1 & 0\\ 0 & -1\end{bmatrix}$$

Equating elements gives

$$-8p_{11} + 6p_{12} = -1, 4p_{12} - 4p_{22} = -1, 2p_{11} - 6p_{12} + 3p_{22} = 0$$

Solving gives $p_{11} = \frac{5}{8}, p_{12} = \frac{2}{3}, p_{22} = \frac{11}{12}$ so that, $\mathbf{P} = \begin{bmatrix}\frac{5}{8} & \frac{2}{3}\\ \frac{2}{3} & \frac{11}{12}\end{bmatrix}$ Principal minors of $\mathbf{P}$ are: $\frac{5}{8} > 0$ and det $\mathbf{P} = (\frac{55}{96} - \frac{4}{9}) > 0$ so $\mathbf{P}$ is positive definite and the system is asymptotically stable

Note that, in this case, we have $V(\mathbf{x}) = \frac{5}{8}x_1^2 + \frac{4}{3}x_1x_2 + \frac{11}{12}x_2^2$ which is positive definite and $\dot{V}(\mathbf{x}) = \frac{5}{4}x_1\dot{x}_1 + \frac{4}{3}\dot{x}_1x_2 + \frac{4}{3}x_1\dot{x}_2 + \frac{11}{6}x_2\dot{x}_2 = -x_1^2 - x_2^2$ which is negative definite.

■ **70** Take tentative Lyapunov function $V(\mathbf{x}) = \mathbf{x}^T\mathbf{P}\mathbf{x}$ giving

$$\begin{aligned}\dot{V}(\mathbf{x}) = \mathbf{x}^T(\mathbf{A}^T\mathbf{P} + \mathbf{P}\mathbf{A})\mathbf{x} = -\mathbf{x}^T\mathbf{Q}\mathbf{x} \text{ where}\\ \mathbf{A}^T\mathbf{P} + \mathbf{P}\mathbf{A} = -\mathbf{Q}\end{aligned} \qquad \text{(i)}$$

Take $\mathbf{Q} = \mathbf{I}$ so that $\dot{V}(\mathbf{x}) = -(x_1^2 + x_2^2)$ which is negative definite. Substituting in (i) gives

$$\begin{bmatrix}-3 & -1\\ 2 & -1\end{bmatrix}\begin{bmatrix}p_{11} & p_{12}\\ p_{12} & p_{22}\end{bmatrix} + \begin{bmatrix}p_{11} & p_{12}\\ p_{12} & p_{22}\end{bmatrix}\begin{bmatrix}-3 & 2\\ -1 & -1\end{bmatrix} = \begin{bmatrix}-1 & 0\\ 0 & -1\end{bmatrix}$$

Equating elements gives

$$-6p_{11} - 2p_{12} = -1, 4p_{12} - 2p_{22} = -1, 2p_{11} - 4p_{12} - p_{22} = 0$$

Solving gives $p_{11} = \frac{7}{40}, p_{12} = -\frac{1}{40}, p_{22} = \frac{18}{40}$ so that $\mathbf{P} = \begin{bmatrix}\frac{7}{40} & -\frac{1}{40}\\ -\frac{1}{40} & \frac{18}{40}\end{bmatrix}$

Principal minors of $\mathbf{P}$ are: $\frac{7}{40} > 0$ and det $\mathbf{P} = \frac{5}{64} > 0$ so $\mathbf{P}$ is positive definite and the system is asymptotically stable.

■ **71** Take tentative Lyapunov function $V(\mathbf{x}) = \mathbf{x}^T\mathbf{P}\mathbf{x}$ giving

$$\dot{V}(\mathbf{x}) = \mathbf{x}^T(\mathbf{A}^T\mathbf{P} + \mathbf{P}\mathbf{A})\mathbf{x} = -\mathbf{x}^T\mathbf{Q}\mathbf{x} \text{ where}$$

$$\mathbf{A}^T\mathbf{P} + \mathbf{P}\mathbf{A} = -\mathbf{Q} \qquad \text{(i)}$$

Take $\mathbf{Q} = \mathbf{I}$ so that $\dot{V}(\mathbf{x}) = -(x_1^2 + x_2^2)$ which is negative definite. Substituting in (i) gives

$$\begin{bmatrix}0 & -a\\ 1 & -b\end{bmatrix}\begin{bmatrix}p_{11} & p_{12}\\ p_{12} & p_{22}\end{bmatrix} + \begin{bmatrix}p_{11} & p_{12}\\ p_{12} & p_{22}\end{bmatrix}\begin{bmatrix}0 & 1\\ -a & -b\end{bmatrix} = \begin{bmatrix}-1 & 0\\ 0 & -1\end{bmatrix}$$

Equating elements gives

$$-8p_{12} = -1, 2p_{12} - 2bp_{22} = -1, p_{11} - bp_{12} - ap_{22} = 0$$

Solving gives $p_{12} = \frac{1}{2a}, p_{22} = \frac{a+1}{2ab}, p_{11} = \frac{b^2+a^2+a}{2ab}$ so that, $\mathbf{P} = \begin{bmatrix} \frac{b^2+a^2+a}{2ab} & \frac{1}{2a} \\ \frac{1}{2a} & \frac{a+1}{2ab} \end{bmatrix}$

For asymptotic stability the principal minors of $\mathbf{P}$ must be positive. Thus,

$$\frac{b^2 + a^2 + a}{2ab} > 0 \quad \text{(ii)}$$

$$\text{and } (b^2 + a^2 + a)(a + 1) > b^2 \quad \text{(iii)}$$

Case 1 $ab > 0$

$$\text{(ii)} \Rightarrow a^2 + b^2 + a > 0 \text{ so (iii)} \Rightarrow a + 1 > \frac{b^2}{b^2 + a^2 + a}$$

$$\Rightarrow a[a^2 + (a + 1)^2] > 0 \Rightarrow a > 0.$$

Since $ab > 0 \Rightarrow b > 0$ it follows that (ii) and (iii) are satisfied if $a, b > 0$
Case 2 $ab < 0$ No solution to (ii) and (iii) in this case.
Thus, system is asymptotically stable when both $a > 0$ and $b > 0$.
Note: This example illustrates the difficulty in interpretating results when using the Lyapunov approach. It is a simple task to confirm this result using the Routh–Hurwitz criterion developed in Section 5.6.2.

- **72(a)**

$$\dot{x}_1 = x_2 \quad \text{(i)}$$

$$\dot{x}_2 = -2x_2 + x_3 \quad \text{(ii)}$$

$$\dot{x}_3 = -kx_1 - x_3 \quad \text{(iii)}$$

If $\dot{V}(\mathbf{x})$ is identically zero then x_3 is identically zero $\Rightarrow x_1$ is identically zero from (iii)

$$\Rightarrow x_2 \text{is identically zero from (i)}$$

Hence $\dot{V}(\mathbf{x})$ is identically zero only at the origin.

(b) $\mathbf{A}^T\mathbf{P} + \mathbf{P}\mathbf{A} = -\mathbf{Q} \Rightarrow$

$$\begin{bmatrix} 0 & 0 & -k \\ 1 & -2 & 0 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} p_{11} & p_{12} & p_{13} \\ p_{12} & p_{22} & p_{23} \\ p_{13} & p_{23} & p_{33} \end{bmatrix} + \begin{bmatrix} p_{11} & p_{12} & p_{13} \\ p_{12} & p_{22} & p_{23} \\ p_{13} & p_{23} & p_{33} \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & -2 & 1 \\ -k & 0 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

Equating elements and solving for the elements of $\boldsymbol{P}$ gives the matrix

$$\mathbf{P} = \begin{bmatrix} \frac{k^2+12k}{12-2k} & \frac{6k}{12-2k} & 0 \\ \frac{6k}{12-2k} & \frac{3k}{12-2k} & \frac{k}{12-2k} \\ 0 & \frac{k}{12-2k} & \frac{6}{12-2k} \end{bmatrix}$$

(c) Principal minors of $\mathbf{P}$ are:

$$\Delta_1 = \frac{k^2+12k}{12-2k} > 0 \text{ if } k > 0 \text{and} (12-2k) > 0 \Rightarrow 0 < k < 6$$

$$\Delta_2 = \left[\frac{k^2+12k}{12-2k}\right]\left[\frac{3k}{12-2k}\right] - \frac{36k^2}{12-2k} = \frac{3k^3}{(12-2k)^2} > 0 \text{ if } k > 0$$

$$\Delta_3 = \frac{(k^2+12k)(8k-k^2)}{(12-2k)^3} - \frac{216k^2}{(12-2k)^3} > 0 \text{if } (6k^3 - k^4) > 0 \Rightarrow 0 < k < 6$$

Thus system asymptotically stable for $0 < k < 6$.

▪ **73** State-space form is

$$\dot{\mathbf{x}} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -k & -a \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \qquad \text{(i)}$$

$$\begin{aligned} \text{Take } V(\mathbf{x}) &= kx_1^2 + (x_2 + ax_1)^2 \text{ then} \\ \dot{V}(\mathbf{x}) &= 2kx_1\dot{x}_1 + 2(x_2 + ax_1)(\dot{x}_2 + a\dot{x}_1) \\ &= 2kx_1(x_2) + 2(x_2 + ax_1)(-kx_1 - ax_2 + ax_1) \text{using (i)} \\ &= -2kax_1^2 \end{aligned}$$

Since $k>0$ and $a>0$ then $\dot{V}(\mathbf{x})$ is negative semidefinite but is not identically zero along any trajectory of (i). Consequently, this choice of Lyapunov function assures asymptotic stability.

Review Exercises 1.13

▪ **1(a)** Eigenvalues given by

$$\begin{vmatrix} -1-\lambda & 6 & 12 \\ 0 & -13-\lambda & 30 \\ 0 & -9 & 20-\lambda \end{vmatrix} = (1+\lambda)[(-13-\lambda)(20-\lambda) + 270] = 0$$

that is, $(1+\lambda)(\lambda-5)(\lambda-2)=0$
so eigenvalues are $\lambda_1=5, \lambda_2=2, \lambda_3=-1$
Eigenvectors are given by corresponding solutions of

$$\begin{bmatrix} -1-\lambda_i & 6 & 12 \\ 0 & -13-\lambda_i & 30 \\ 0 & -9 & 20-\lambda_i \end{bmatrix} \begin{bmatrix} e_{i1} \\ e_{i2} \\ e_{i3} \end{bmatrix} = 0$$

When $i=1, \lambda_i=5$ and solution given by

$$\frac{e_{11}}{198} = \frac{-e_{12}}{-90} = \frac{e_{13}}{54} = \beta_1$$

so $\mathbf{e}_1 = [11\ 5\ 3]^T$
When $i=2, \lambda_i=2$ and solution given by

$$\frac{e_{21}}{216} = \frac{-e_{22}}{-54} = \frac{e_{23}}{27} = \beta_2$$

so $\mathbf{e}_2 = [8\ 2\ 1]^T$
When $i=3, \lambda_i=-1$ and solution given by

$$\frac{e_{31}}{1} = \frac{-e_{32}}{0} = \frac{e_{33}}{0} = \beta_3$$

so $\mathbf{e}_3 = [1\ 0\ 0]^T$

1(b) Eigenvalues given by

$$\begin{vmatrix} 2-\lambda & 0 & 1 \\ -1 & 4-\lambda & -1 \\ -1 & 2 & 0-\lambda \end{vmatrix} = \begin{vmatrix} 4-\lambda & -1 \\ 2 & -\lambda \end{vmatrix} + \begin{vmatrix} -1 & 4-\lambda \\ -1 & 2 \end{vmatrix} = 0$$

$$\begin{aligned} \text{that is, } 0 &= (2-\lambda)[(4-\lambda)(-\lambda)+2] + [-2+(4-\lambda)] \\ &= (2-\lambda)(\lambda^2-4\lambda+3) = (2-\lambda)(\lambda-3)(\lambda-1) = 0 \end{aligned}$$

so eigenvalues are

$$\lambda_1=3, \lambda_2=2, \lambda_3=1$$

Eigenvectors are given by the corresponding solutions of

$$\begin{aligned} (2-\lambda_i)e_{i1} + 0e_{i2} + e_{i3} &= 0 \\ -e_{i1} + (4-\lambda_i)e_{i2} - e_{i3} &= 0 \\ -e_{i1} + 2e_{i2} - \lambda_i e_{i3} &= 0 \end{aligned}$$

Taking $i = 1, 2, 3$ gives the eigenvectors as

$$\mathbf{e}_1 = [1\ 2\ 1]^T, \mathbf{e}_2 = [2\ 1\ 0]^T, \mathbf{e}_3 = [1\ 0\ -1]^T$$

1(c) Eigenvalues given by

$$\begin{vmatrix} 1-\lambda & -1 & 0 \\ -1 & 2-\lambda & -1 \\ 0 & -1 & 1-\lambda \end{vmatrix} \quad R_1 + (\underline{\underline{R_2}} + R_3) \quad \begin{vmatrix} -\lambda & -\lambda & -\lambda \\ -1 & 2-\lambda & -1 \\ 0 & -1 & 1-\lambda \end{vmatrix} = 0$$

$$\text{that is, } \lambda \begin{vmatrix} -1 & -1 & -1 \\ -1 & 2-\lambda & -1 \\ 0 & -1 & 1-\lambda \end{vmatrix} = \lambda \begin{vmatrix} -1 & 0 & 0 \\ -1 & 3-\lambda & 0 \\ 0 & -1 & 1-\lambda \end{vmatrix} = \lambda(3-\lambda)(1-\lambda) = 0$$

so eigenvalues are $\lambda_1 = 3, \lambda_2 = 1, \lambda_3 = 0$

Eigenvalues are given by the corresponding solutions of

$$(1-\lambda_i)e_{i1} - e_{i2} - 0e_{i3} = 0$$

$$-e_{i1} + (2-\lambda_i)e_{i2} - e_{i3} = 0$$

$$0e_{i1} - e_{i2} + (1-\lambda_i)e_{i3} = 0$$

Taking $i = 1, 2, 3$ gives the eigenvectors as

$$\mathbf{e}_1 = [1\ -2\ 1]^T, \mathbf{e} = [1\ 0\ -1]^T, \mathbf{e}_3 = [1\ 1\ 1]^T$$

■ **2** Principal stress values (eigenvalues) given by

$$\begin{vmatrix} 3-\lambda & 2 & 1 \\ 2 & 3-\lambda & 1 \\ 1 & 1 & 4-\lambda \end{vmatrix} \quad R_1 + (\underline{\underline{R_2}} + R_3) \begin{vmatrix} 6-\lambda & 6-\lambda & 6-\lambda \\ 2 & 3-\lambda & 1 \\ 1 & 1 & 4-\lambda \end{vmatrix}$$

$$= (6-\lambda) \begin{vmatrix} 1 & 1 & 1 \\ 2 & 3-\lambda & 1 \\ 1 & 1 & 4-\lambda \end{vmatrix} = 0$$

$$\text{that is, } (6-\lambda) \begin{vmatrix} 1 & 0 & 0 \\ 2 & 1-\lambda & -1 \\ 1 & 0 & 3-\lambda \end{vmatrix} = (6-\lambda)(1-\lambda)(3-\lambda) = 0$$

so the principal stress values are $\lambda_1 = 6, \lambda_2 = 3, \lambda_3 = 1$.

Corresponding principal stress direction $\mathbf{e}_1, \mathbf{e}_2$ and $\mathbf{e}_3$ are given by the solutions of

$$(3-\lambda_i)e_{i1} + 2e_{i2} + e_{i3} = 0$$
$$2e_{i1} + (3-\lambda_i)e_{i2} + e_{i3} = 0$$
$$e_{i1} + e_{i2} + (4-\lambda_i)e_{i3} = 0$$

Taking $i = 1, 2, 3$ gives the principal stress direction as

$$\mathbf{e}_1 = [1\ 1\ 1]^T,\ \mathbf{e}_2 = [1\ 1\ -2]^T,\ \mathbf{e}_3 = [1\ -1\ 0]^T$$

It is readily shown that $\mathbf{e}_1^T\mathbf{e}_2 = \mathbf{e}_1^T\mathbf{e}_3 = \mathbf{e}_2^T\mathbf{e}_3 = 0$ so that the principal stress directions are mutually orthogonal.

■ **3** Since $[1\ 0\ 1]^T$ is an eigenvector of $\mathbf{A}$

$$\begin{bmatrix} 2 & -1 & 0 \\ -1 & 3 & b \\ 0 & b & c \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \lambda \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

so $2 = \lambda, -1 + b = 0, c = \lambda$
giving $b = 1$ and $c = 2$.
Taking these values $\mathbf{A}$ has eigenvalues given by

$$\begin{vmatrix} 2-\lambda & -1 & 0 \\ -1 & 3-\lambda & 1 \\ 0 & 1 & 2-\lambda \end{vmatrix} = (2-\lambda)\begin{vmatrix} 3-\lambda & 1 \\ 1 & 2-\lambda \end{vmatrix} - (2-\lambda)$$
$$= (2-\lambda)(\lambda-1)(\lambda-4) = 0$$

that is, eigenvalues are $\lambda_1 = 4, \lambda_2 = 2, \lambda_3 = 1$
Corresponding eigenvalues are given by the solutions of

$$(2-\lambda_i)e_{i1} - e_{i2} + 0e_{i3} = 0$$
$$-e_{i1} + (3-\lambda_i)e_{i2} + e_{i3} = 0$$
$$0e_{i1} + e_{i2} + (2-\lambda_i)e_{i3} = 0$$

Taking $i = 1, 2, 3$ gives the eigenvectors as

$$\mathbf{e}_1 = [1\ -2\ -1]^T, \mathbf{e}_2 = [1\ 0\ 1]^T, \mathbf{e}_3 = [1\ 1\ -1]^T$$

■ **4** The three Gerschgorin circles are

$$| \lambda - 4 | = | -1 | + | 0 | = 1$$
$$| \lambda - 4 | = | -1 | + | -1 | = 2$$
$$| \lambda - 4 | = 1$$

Thus, $| \lambda - 4 | \leq 1$ and $| \lambda - 4 | \leq 2$ so $| \lambda - 4 | \leq 2$ or $2 \leq \lambda \leq 6$.
Taking $\mathbf{x}^{(o)} = [-1\ 1\ -1]^T$ iterations using the power method may be tabulated as follows

Iteration k	**0**	**1**	**2**	**3**	**4**	**5**	**6**
	-1	-0.833	-0.765	-0.734	-0.720	-0.713	-0.710
$\mathbf{x}^{(k)}$	1	1	1	1	1	1	1
	-1	-0.833	-0.765	-0.734	-0.720	-0.713	-0.710
	-5	-4.332	-4.060	-3.936	-3.88	-3.852	
$\mathbf{A}\ \mathbf{x}^{(k)}$	6	5.666	5.530	5.468	5.44	5.426	
	-5	-4.332	-4.060	-3.936	3.88	-3.852	
$\lambda \simeq$	6	5.666	5.530	5.468	5.44	5.426	

Thus, correct to one decimal place the dominant eigenvalue is $\lambda = 5.4$

■ **5(a)** Taking $\boldsymbol{x}^{(o)} = [1\ 1\ 1]^7$ iterations may be tabulated as follows

Iteration k	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**
	1	0.800	0.745	0.728	0.722	0.720	0.719	0.719
$\mathbf{x}^{(k)}$	1	0.900	0.862	0.847	0.841	0.838	0.837	0.837
	1	1	1	1	1	1	1	1
	4	3.500	3.352	3.303	3.285	3.278	3.275	
$\mathbf{A}\ \mathbf{x}^{(k)}$	4.5	4.050	3.900	3.846	3.825	3.815	3.812	
	5	4.700	4.607	4.575	4.563	4.558	4.556	
$\lambda \simeq$	5	4.700	4.607	4.575	4.563	4.558	4.556	

Thus, estimate of dominant eigenvalues is $\lambda \simeq 4.56$ with associated eigenvector $\mathbf{x} = [0.72\ 0.84\ 1]^T$

5(b) $\sum_{i=1}^{3} \lambda_i =$ trace $\mathbf{A} \Rightarrow 7.5 = 4.56 + 1.19 + \lambda_3 \Rightarrow \lambda_3 = 1.75$

5(c) (i) $\det \mathbf{A} = \prod_{i=1}^{3} \lambda_i = 9.50$ so $\mathbf{A}^{-1}$ exists and has eigenvalues

$$\frac{1}{1.19}, \frac{1}{1.75}, \frac{1}{4.56}$$

so power method will generate the eigenvalue 1.19 corresponding to $\mathbf{A}$.

(ii) $\mathbf{A} - 3\mathbf{I}$ has eigenvalues

$$1.19 - 3,\ 1.75 - 3,\ 4.56 - 3$$

$$\text{that is, } -1.91,\ -1.25,\ 1.56$$

so applying the power method on $\mathbf{A} - 3\mathbf{I}$ generates the eigenvalues corresponding to 1.75 of $\mathbf{A}$.

■ **6** $\dot{x} = \alpha\lambda e^{\lambda t},\ \dot{y} = \beta\lambda e^{\lambda t},\ \dot{z} = \gamma\lambda e^{\lambda t}$ so the differential equations become

$$\begin{aligned} \alpha\lambda e^{\lambda t} &= 4\alpha e^{\lambda t} + \beta e^{\lambda t} + \gamma e^{\lambda t} \\ \beta\lambda e^{\lambda t} &= 2\alpha e^{\lambda t} + 5\beta e^{\lambda t} + 4\gamma e^{\lambda t} \\ \gamma\lambda e^{\lambda t} &= -\alpha e^{\lambda t} - \beta e^{\lambda t} \end{aligned}$$

Provided $e^{\lambda t} \neq 0$ (i.e. non-trivial solution) we have the eigenvalue problem

$$\begin{bmatrix} 4 & 1 & 1 \\ 2 & 5 & 4 \\ -1 & -1 & 0 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = \lambda \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix}$$

Eigenvalues given by

$$\begin{vmatrix} 4-\lambda & 1 & 1 \\ 2 & 5-\lambda & 4 \\ -1 & -1 & 0 \end{vmatrix} \underline{\underline{C_2 - C_3}} \begin{vmatrix} 4-\lambda & 0 & 1 \\ 2 & 1-\lambda & 4 \\ -1 & \lambda - 1 & -\lambda \end{vmatrix} = (\lambda - 1) \begin{vmatrix} 4-\lambda & 0 & 1 \\ 2 & -1 & 4 \\ -1 & 1 & -\lambda \end{vmatrix}$$

$$= -(\lambda - 1)(\lambda - 5)(\lambda - 3)$$

so its eigenvalues are 5, 3 and 1.
When $\lambda = 1$ the corresponding eigenvector is given by

$$\begin{aligned} 3e_{11} + e_{12} + e_{13} &= 0 \\ 2e_{11} + 4e_{12} + 4e_{13} &= 0 \\ -e_{11} - e_{12} - e_{13} &= 0 \end{aligned}$$

having solution $\dfrac{e_{11}}{0} = \dfrac{-e_{12}}{2} = \dfrac{e_{13}}{2} = \beta_1$
Thus, corresponding eigenvector is $\beta[0 \ -1 \ 1]^T$

■ **7** Eigenvalues are given by

$$| \mathbf{A} - \lambda\mathbf{I} | = \begin{vmatrix} 8-\lambda & -8 & -2 \\ 4 & -3-\lambda & -2 \\ 3 & -4 & 1-\lambda \end{vmatrix} = 0$$

Row 1 $-$ (Row 2 $+$ Row 3) gives

$$| \mathbf{A} - \lambda\mathbf{I} | = \begin{vmatrix} 1-\lambda & -1+\lambda & -1+\lambda \\ 4 & -3-\lambda & -2 \\ 3 & -4 & 1-\lambda \end{vmatrix} = (1-\lambda)\begin{vmatrix} 1 & -1 & -1 \\ 4 & -3-\lambda & -2 \\ 3 & -4 & 1-\lambda \end{vmatrix}$$

$$= (1-\lambda)\begin{vmatrix} 1 & 0 & 0 \\ 4 & 1-\lambda & 2 \\ 3 & -1 & 4-\lambda \end{vmatrix} = (1-\lambda)[(1-\lambda)(4-\lambda)+2]$$

$$= (1-\lambda)(\lambda-2)(\lambda-3)$$

Thus, eigenvalues are $\lambda_1 = 3, \lambda_2 = 2, \lambda_3 = 1$.
Corresponding eigenvectors are given by

$$(8-\lambda)e_{i1} - 8e_{i2} - 2e_{i3} = 0$$

$$4e_{i1} - (3+\lambda)e_{i2} - 2e_{i3} = 0$$

$$3e_{i1} - 4e_{i2} + (1-\lambda)e_{i3} = 0$$

When $i = 1, \lambda_i = \lambda_1 = 3$ and solution given by

$$\frac{e_{11}}{4} = \frac{-e_{12}}{-2} = \frac{e_{13}}{2} = \beta_1$$

so a corresponding eigenvector is $\mathbf{e}_1 = [2 \ 1 \ 1]^T$.
When $i = 2, \lambda_i = \lambda_2 = 2$ and solution given by

$$\frac{e_{21}}{-3} = \frac{-e_{22}}{2} = \frac{e_{23}}{-1} = \beta_2$$

so a corresponding eigenvector is $\mathbf{e}_2 = [3 \ 2 \ 1]^T$.
When $i = 3, \lambda_i = \lambda_3 = 1$ and solution given by

$$\frac{e_{31}}{-8} = \frac{-e_{32}}{6} = \frac{e_{33}}{-4} = \beta_3$$

so a corresponding eigenvector is $\mathbf{e}_3 = [4\ 3\ 2]^T$.
Corresponding modal and spectral matrices are

$$\mathbf{M} = \begin{bmatrix} 2 & 3 & 4 \\ 1 & 2 & 3 \\ 1 & 1 & 2 \end{bmatrix} \text{ and } \mathbf{\Lambda} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

$\mathbf{M}^{-1} = \begin{bmatrix} 1 & -2 & 1 \\ 1 & 0 & -2 \\ -1 & 1 & 1 \end{bmatrix}$ and matrix multiplication confirms $\mathbf{M}^{-1}\ \mathbf{A}\ \mathbf{M} = \mathbf{\Lambda}$

■ **8** Eigenvectors of $\mathbf{A}$ are given by

$$\begin{vmatrix} 1-\lambda & 0 & -4 \\ 0 & 5-\lambda & 4 \\ -4 & 4 & 3-\lambda \end{vmatrix} = 0$$

that is, $\lambda^3 - 9\lambda^2 - 9\lambda + 81 = (\lambda - 9)(\lambda - 3)(\lambda + 3) = 0$
so the eigenvalues are $\lambda_1 = 9, \lambda_2 = 3$ and $\lambda_3 = -3$.
The eigenvectors are given by the corresponding solutions of

$$(1-\lambda_i)e_{i1} + 0e_{i2} - 4e_{i3} = 0$$

$$0e_{i1} + (5-\lambda_i)e_{i2} + 4e_{i3} = 0$$

$$-4e_{i1} + 4e_{i2} + (3-\lambda_i)e_{i3} = 0$$

Taking $i = 1, 2, 3$ the normalized eigenvectors are given by

$$\hat{\mathbf{e}}_1 = [\tfrac{1}{3}\ \tfrac{-2}{3}\ \tfrac{-2}{3}]^T, \hat{\mathbf{e}}_2 = [\tfrac{2}{3}\ \tfrac{2}{3}\ \tfrac{-1}{3}]^T, \hat{\mathbf{e}}_3 = [\tfrac{2}{3}\ \tfrac{-1}{3}\ \tfrac{2}{3}]^T$$

The normalised modal matrix

$$\hat{\mathbf{M}} = \frac{1}{3}\begin{bmatrix} 1 & 2 & 2 \\ -2 & 2 & -1 \\ -2 & -1 & 2 \end{bmatrix}$$

so

$$\hat{\mathbf{M}}^T\mathbf{A}\ \hat{\mathbf{M}} = \frac{1}{9}\begin{bmatrix} 1 & -2 & -2 \\ 2 & 2 & -1 \\ 2 & -1 & 2 \end{bmatrix}\begin{bmatrix} 1 & 0 & -4 \\ 0 & 5 & 4 \\ -4 & 4 & 3 \end{bmatrix}\begin{bmatrix} 1 & 2 & 2 \\ -2 & 2 & -1 \\ -2 & -1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 9 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -3 \end{bmatrix} = \mathbf{\Lambda}$$

■ **9** $\dot{\mathbf{N}} = \begin{bmatrix} -6 & 0 & 0 & 0 \\ 6 & -4 & 0 & 0 \\ 0 & 4 & -2 & 0 \\ 0 & 0 & 2 & 0 \end{bmatrix} \mathbf{N},\ \mathbf{N} = [N_1\ N_2\ N_3\ N_4]^T$

Since the matrix $\mathbf{A}$ is a triangular matrix its eigenvalues are the diagonal elements. Thus, the eigenvalues are

$$\lambda_1 = -6,\ \lambda_2 = -4, \lambda_3 = -2, \lambda_4 = 0$$

The eigenvectors are the corresponding solutions of

$$\begin{aligned}
(-6 - \lambda_i)e_{i1} + 0e_{i2} + 0e_{i3} + 0e_{i4} &= 0 \\
6e_{i1} + (-4 - \lambda_i)e_{i2} + 0e_{i3} + 0e_{i4} &= 0 \\
0e_{i1} + 4e_{i2} + (-2 - \lambda_i)e_{i3} + 0e_{i4} &= 0 \\
0e_{i1} + 0e_{i2} + 2e_{i3} - \lambda_i e_{i4} &= 0
\end{aligned}$$

Taking $i = 1, 2, 3, 4$ and solving gives the eigenvectors as

$$\mathbf{e}_1 = [1\ \ -3\ 3\ \ -1]^T,\ \mathbf{e}_2 = [0\ 1\ \ -2\ 1]^T$$
$$\mathbf{e}_3 = [0\ 0\ 1\ \ -1]^T,\ \mathbf{e}_4 = [0\ 0\ 0\ 1]^T$$

Thus, spectral form of solution to the equation is

$$\mathbf{N} = \alpha e^{-6t}\mathbf{e}_1 + \beta e^{-4t}\mathbf{e}_2 + \gamma e^{-2t}\mathbf{e}_3 + \delta\mathbf{e}_4$$

Using the given initial conditions at $t = 0$ we have

$$\begin{bmatrix} C \\ 0 \\ 0 \\ 0 \end{bmatrix} = \alpha \begin{bmatrix} 1 \\ -3 \\ 3 \\ -1 \end{bmatrix} + \beta \begin{bmatrix} 0 \\ 1 \\ -2 \\ 1 \end{bmatrix} + \gamma \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix} + \delta \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

so $C = \alpha,\ 0 = -3\alpha + \beta,\ 0 = 3\alpha - 2\beta + \gamma,\ 0 = -\alpha + \beta - \gamma + \delta$ which may be solved for α, β, γ and δ to give

$$\alpha = C,\ \beta = 3C,\ \gamma = 3C,\ \delta = C$$

Hence,

$$\begin{aligned}
N_4 &= -\alpha e^{-6t} + \beta e^{-4t} - \gamma e^{-2t} + \delta \\
&= -Ce^{-6t} + 3Ce^{-4t} - 3Ce^{-2t} + C
\end{aligned}$$

■ **10(a)**

(i) Characteristic equation of $\mathbf{A}$ is $\lambda^2 - 3\lambda + 2 = 0$ so by the Cayley–Hamilton theorem

$$\mathbf{A}^2 = 3\mathbf{A} - 2\mathbf{I} = \begin{bmatrix} 4 & 0 \\ 3 & 1 \end{bmatrix}$$

$$\mathbf{A}^3 = 3(3\mathbf{A} - 2\mathbf{I}) - 2\mathbf{A} = 7\mathbf{A} - 6\mathbf{I} = \begin{bmatrix} 8 & 0 \\ 7 & 1 \end{bmatrix}$$

$$\mathbf{A}^4 = 7(3\mathbf{A} - 2\mathbf{I}) - 6\mathbf{A} = 15\mathbf{A} - 14\mathbf{I} = \begin{bmatrix} 16 & 0 \\ 15 & 1 \end{bmatrix}$$

$$\mathbf{A}^5 = 15(3\mathbf{A} - 2\mathbf{I}|) - 14\mathbf{A} = 31\mathbf{A} - 30\mathbf{I} = \begin{bmatrix} 32 & 0 \\ 31 & 1 \end{bmatrix}$$

$$\mathbf{A}^6 = 31(3\mathbf{A} - 2\mathbf{I}) - 30\mathbf{A} = 63\mathbf{A} - 62\mathbf{I} = \begin{bmatrix} 64 & 0 \\ 63 & 1 \end{bmatrix}$$

$$\mathbf{A}^7 = 63(3\mathbf{A} - 2\mathbf{I}) - 62\mathbf{A} = 127\mathbf{A} - 126\mathbf{I} = \begin{bmatrix} 128 & 0 \\ 127 & 1 \end{bmatrix}$$

Thus, $\mathbf{A}^7 - 3\mathbf{A}^6 + \mathbf{A}^4 + 3\mathbf{A}^3 - 2\mathbf{A}^2 + 3\mathbf{I} = \begin{bmatrix} -29 & 0 \\ -32 & 3 \end{bmatrix}$

(ii) Eigenvalues of $\mathbf{A}$ are $\lambda_1 = 2, \lambda_2 = 1$. Thus,

$$\mathbf{A}^k = \alpha_0\mathbf{I} + \alpha_1\mathbf{A} \text{ where } \alpha_0 \text{ and } \alpha_1 \text{ satisfy}$$

$$2^k = \alpha_0 + 2\alpha_1, 1 = \alpha_0 + \alpha_1$$

$$\alpha_1 = 2^k - 1,\ \alpha_0 = 2 - 2^k$$

Thus, $\mathbf{A}^k = \begin{bmatrix} \alpha_0 + 2\alpha_1 & 0 \\ \alpha_1 & \alpha_0 + \alpha_1 \end{bmatrix} = \begin{bmatrix} 2^k & 0 \\ 2^k - 1 & 1 \end{bmatrix}$

10(b) Eigenvalues of $\mathbf{A}$ are $\lambda_1 = -2, \lambda_2 = 0$. Thus,

$$e^{\mathbf{A}t} = \alpha_0\mathbf{I} + \alpha_1\mathbf{A} \text{ where } \alpha_0 \text{ and } \alpha_1 \text{ satisfy}$$

$$e^{-2t} = \alpha_0 - 2\alpha_1,\ 1 = \alpha_0 \ \Rightarrow \alpha_0 = 1, \alpha_1 = \frac{1}{2}(1 - e^{-2t})$$

Thus, $e^{\mathbf{A}t} = \begin{bmatrix} \alpha_0 & \alpha_1 \\ 0 & \alpha_0 - 2\alpha_1 \end{bmatrix} = \begin{bmatrix} 1 & \frac{1}{2}(1 - e^{-2t}) \\ 0 & e^{-2t} \end{bmatrix}$

■ **11** The matrix $\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix}$ has the single eigenvalue $\lambda = 1$ (multiplicity 3)

$(\mathbf{A} - \mathbf{I}) = \begin{bmatrix} 0 & 2 & 3 \\ 0 & 0 & 4 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ is of rank 2 so has nullity $3 - 2 = 1$ indicating that there is only one eigenvector corresponding to $\lambda = 1$.
This is readily determined as

$$\mathbf{e}_1 = [1\ 0\ 0]^T$$

The corresponding Jordan canonical form comprises a single block so

$$\mathbf{J} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

Taking $\mathbf{T} = \mathbf{A} - \mathbf{I}$ the triad of vectors (including generalized eigenvectors) has the form $\{\mathbf{T}^2\omega,\ \mathbf{T}\ \omega, \omega\}$ with $\mathbf{T}^2\omega = \mathbf{e}_1$. Since $\mathbf{T}^2 = \begin{bmatrix} 0 & 0 & 8 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, we may take $\omega = [0\ 0\ \frac{1}{8}]^T$. Then, $\mathbf{T}\ \omega = [\frac{2}{8}\ \frac{1}{8}\ 0]^T$. Thus, the triad of vectors is

$$\mathbf{e}_1 = [1\ 0\ 0]^T,\ \mathbf{e}_1^* = [\tfrac{3}{8}\ \tfrac{1}{2}\ 0]^T,\ \mathbf{e}_1^{**} = [0\ 0\ \tfrac{1}{8}]^T$$

The corresponding modal matrix is

$$\mathbf{M} = \begin{bmatrix} 1 & \frac{3}{8} & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{8} \end{bmatrix}$$

$$\mathbf{M}^{-1} = 16 \begin{bmatrix} \frac{1}{16} & -\frac{3}{64} & 0 \\ 0 & \frac{1}{8} & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix}$$ and by matrix multiplication

$$\mathbf{M}^{-1}\ \mathbf{A}\ \mathbf{M} = 16 \begin{bmatrix} \frac{1}{16} & -\frac{3}{64} & 0 \\ 0 & \frac{1}{8} & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & \frac{3}{8} & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{8} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = \mathbf{J}$$

■ **12** Substituting $x = X\cos\omega t,\ y = Y\cos\omega t,\ z = Z\cos\omega t$ gives

$$-\omega^2 X = -2X + Y$$
$$-\omega^2 Y = X - 2Y + Z$$
$$-\omega^2 Z = Y - 2Z$$

or taking $\lambda = \omega^2$

$$(\lambda - 2)X + Y = 0$$
$$X + (\lambda - 2)Y + Z = 0$$
$$Y + (\lambda - 2)Z = 0$$

For non-trivial solution

$$\begin{vmatrix} \lambda - 2 & 1 & 0 \\ 1 & \lambda - 2 & 1 \\ 0 & 1 & \lambda - 2 \end{vmatrix} = 0$$

$$\text{that is, } (\lambda - 2)[(\lambda - 2)^2 - 1] - (\lambda - 2) = 0$$
$$(\lambda - 2)(\lambda^2 - 4\lambda + 2) = 0$$
$$\text{so } \lambda = 2 \text{ or } \lambda = 2 \pm \sqrt{2}$$

When $\lambda = 2$, $Y = 0$ and $X = -Z$ so $X : Y : Z = 1 : 0 : -1$
When $\lambda = 2 + \sqrt{2}$, $X = Z$ and $Y = -\sqrt{2}X$ so $X : Y : Z = 1 : -\sqrt{2} : 1$
When $\lambda = 2 - \sqrt{2}$, $X = Z$ and $Y = \sqrt{2}X$ so $X : Y : Z = 1 : \sqrt{2} : 1$

■ **13** In each section $\mathbf{A}$ denotes the matrix of the quadratic form.

13(a) $\mathbf{A} = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 1 & -1 \\ 0 & -1 & 2 \end{bmatrix}$ has principal minors of 2, $\begin{vmatrix} 2 & -1 \\ -1 & 1 \end{vmatrix} = 1$ and
$\det \mathbf{A} = 0$
so by Sylvester's condition (c) the quadratic form is positive-semidefinite.

13(b) $\mathbf{A} = \begin{bmatrix} 3 & -2 & -2 \\ -2 & 7 & 0 \\ -2 & 0 & 2 \end{bmatrix}$ has principal minors of 3, $\begin{vmatrix} 3 & -2 \\ -2 & 7 \end{vmatrix} = 17$ and
$\det \mathbf{A} = 6$
so by Sylvester's condition (a) the quadratic form is positive-definite.

13(c) $\mathbf{A} = \begin{bmatrix} 16 & 16 & 16 \\ 16 & 36 & 8 \\ 16 & 8 & 17 \end{bmatrix}$ has principal minors of 16, $\begin{vmatrix} 16 & 16 \\ 16 & 36 \end{vmatrix} = 320$ and $\det \mathbf{A} = -704$

so none of Sylvester's conditions are satisfied and the quadratic form is indefinite.

13(d) $\mathbf{A} = \begin{bmatrix} -21 & 15 & -6 \\ 15 & -11 & 4 \\ -6 & 4 & -2 \end{bmatrix}$ has principal minors of -21, $\begin{vmatrix} -21 & 15 \\ 15 & -11 \end{vmatrix} = 6$ and $\det \mathbf{A} = 0$

so by Sylvester's condition (d) the quadratic form is negative-semidefinite.

13(e) $\mathbf{A} = \begin{bmatrix} -1 & 1 & 1 \\ 1 & -3 & 1 \\ 1 & 1 & -5 \end{bmatrix}$ has principal minors of -1, $\begin{vmatrix} -1 & 1 \\ 1 & -3 \end{vmatrix} = 2$ and $\det \mathbf{A} = -4$ so by Sylvester's condition (b) the quadratic form is negative-definite.

■ **14** $\mathbf{A}\,\mathbf{e}_1 = \begin{bmatrix} \frac{7}{2} & -\frac{1}{2} & -\frac{1}{2} \\ 4 & -1 & 0 \\ -\frac{3}{2} & \frac{3}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

Hence, $\mathbf{e}_1 = [1\ 2\ 3]^T$ is an eigenvector with $\lambda_1 = 1$ the corresponding eigenvalue. Eigenvalues are given by

$$0 = \begin{vmatrix} -\frac{7}{2} - \lambda & -\frac{1}{2} & -\frac{1}{2} \\ 4 & -1-\lambda & 0 \\ -\frac{3}{2} & \frac{3}{2} & \frac{1}{2} - \lambda \end{vmatrix} = -\lambda^3 + 3\lambda^2 + \lambda - 3$$

$$= (\lambda - 1)(\lambda^2 + 2\lambda + 3)$$

$$= -(\lambda - 1)(\lambda - 3)(\lambda + 1)$$

so the other two eigenvalues are $\lambda_2 = 3, \lambda_3 = -1$.

Corresponding eigenvectors are the solutions of

$$(-\tfrac{7}{2} - \lambda_i)e_{i1} - \tfrac{1}{2}e_{i2} - \tfrac{1}{2}e_{i3} = 0$$

$$4e_{i1} - (1 + \lambda_i)e_{i2} + 0e_{i3} = 0$$

$$-\tfrac{3}{2}e_{i1} + \tfrac{3}{2}e_{i2} + (\tfrac{1}{2} - \lambda_i)e_{i3} = 0$$

Taking $i = 2, 3$ gives the eigenvectors as

$$\mathbf{e}_2 = [1\ 1\ 0]^T,\ \mathbf{e}_3 = [0\ -1\ 1]^T$$

The differential equations can be written in the vector–matrix form

$$\dot{\mathbf{x}} = \mathbf{A}\ \mathbf{x}\ ,\ \mathbf{x} = [x\ y\ z]^T$$

so, in special form, the general solution is

$$\begin{aligned}\mathbf{x} &= \alpha e^{\lambda_1 t}\mathbf{e}_1 + \beta e^{\lambda_2 t}\mathbf{e}_2 + \gamma e_{\lambda_3 t}\mathbf{e}_3 \\ &= \alpha e^t \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + \beta e^{3t} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + \gamma e^{-t} \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}\end{aligned}$$

With $x(0) = 2,\ y(0) = 4,\ z(0) = 6$ we have

$$\alpha = 2,\ \beta = 0,\ \gamma = 0$$

so

$$\mathbf{x} = 2e^t \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

that is, $x = 2e^t,\ y = 4e^t,\ z = 6e^t$.

- **15(a)**

$$\mathbf{A}\mathbf{A}^T = \begin{bmatrix} 1.2 & 0.9 & -4 \\ 1.6 & 1.2 & 3 \end{bmatrix} \begin{bmatrix} 1.2 & 1.6 \\ 0.9 & 1.2 \\ -4 & 3 \end{bmatrix} = \begin{bmatrix} 18.25 & -9 \\ -9 & 13 \end{bmatrix}$$

Eigenvalues λ_{i} given by

$$\begin{aligned}(18.25 - \lambda)(13 - \lambda) - 81 = 0 &\Rightarrow (\lambda - 25)(\lambda - 6.25) = 0 \\ &\Rightarrow \lambda_1 = 25, \lambda_2 = 6.25\end{aligned}$$

having corresponding eigenvectors

$$\begin{aligned}\mathbf{u}_1 &= [-4\quad 3]^T \Rightarrow \hat{\mathbf{u}}_1 = [-\tfrac{4}{5}\quad \tfrac{3}{5}]^T \\ \mathbf{u}_2 &= [3\quad 4]^T \Rightarrow \hat{\mathbf{u}}_2 = [\tfrac{3}{5}\quad \tfrac{4}{5}]\end{aligned}$$

leading to the orthogonal matrix

$$\hat{\mathbf{U}} = \begin{bmatrix} -\frac{4}{5} & \frac{3}{5} \\ \frac{3}{5} & \frac{4}{5} \end{bmatrix}$$

$$\mathbf{A}^T\mathbf{A} = \begin{bmatrix} 1.2 & 1.6 \\ 0.9 & 1.2 \\ -4 & 3 \end{bmatrix} \begin{bmatrix} 1.2 & 0.9 & -4 \\ 1.6 & 1.2 & 3 \end{bmatrix} = \begin{bmatrix} 4 & 3 & 0 \\ 3 & 2.25 & 0 \\ 0 & 0 & 25 \end{bmatrix}$$

Eigenvalues μ_{i} given by

$$(25 - \mu)\left[(4 - \mu)(2.25 - \mu) - 9\right] = 0 \Rightarrow (25 - \mu)\mu(\mu - 6.25) = 0$$

$$\Rightarrow \mu_1 = 25, \mu_2 = 6.25, \mu_3 = 0$$

with corresponding eigenvalues

$$\begin{aligned} \mathbf{v}_1 &= \hat{\mathbf{v}}_1 = [\,0 \quad 0 \quad 1\,]^T \\ \mathbf{v}_2 &= [\,4 \quad 3 \quad 0\,]^T \Rightarrow \hat{\mathbf{v}}_2 = [\,\tfrac{4}{5} \quad \tfrac{3}{5} \quad 0\,]^T \\ \mathbf{v}_3 &= [\,-3 \quad 4 \quad 0\,]^T \Rightarrow \hat{\mathbf{v}}_3 = [\,-\tfrac{3}{5} \quad \tfrac{4}{5} \quad 0\,]^T \end{aligned}$$

leading to the orthogonal matrix

$$\hat{\mathbf{V}} = \begin{bmatrix} 0 & \frac{4}{5} & -\frac{3}{5} \\ 0 & \frac{3}{5} & \frac{4}{5} \\ 1 & 0 & 0 \end{bmatrix}$$

The singular values of $\boldsymbol{A}$ are $\sigma_1 = \sqrt{25} = 5$ and $\sigma_2 = \sqrt{6.25} = 2.5$ so that $\boldsymbol{\Sigma} = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 2.5 & 0 \end{bmatrix}$ giving the SVD form of $\boldsymbol{A}$ as

$$\mathbf{A} = \hat{\mathbf{U}}\boldsymbol{\Sigma}\hat{\mathbf{V}}^T = \begin{bmatrix} -0.8 & 0.6 \\ 0.6 & 0.8 \end{bmatrix} \begin{bmatrix} 5 & 0 & 0 \\ 0 & 2.5 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 0.8 & 0.6 & 0 \\ -0.6 & 0.8 & 0 \end{bmatrix}$$

(Direct multiplication confirms $\mathbf{A} = \begin{bmatrix} 1.2 & 0.9 & -4 \\ 1.6 & 1.2 & 3 \end{bmatrix}$)

(b) $\mathbf{A}^\dagger = \hat{\mathbf{V}}\mathbf{\Sigma}^*\hat{\mathbf{U}}^T = \begin{bmatrix} 0 & \frac{4}{5} & -\frac{3}{5} \\ 0 & \frac{3}{5} & \frac{4}{5} \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{5} & 0 \\ 0 & \frac{2}{5} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} -\frac{4}{5} & \frac{3}{5} \\ \frac{3}{5} & \frac{4}{5} \end{bmatrix} = \frac{1}{125}\begin{bmatrix} 24 & 32 \\ 18 & 24 \\ -20 & 15 \end{bmatrix}$

$$= \begin{bmatrix} 0.192 & 0.256 \\ 0.144 & 0.192 \\ -0.16 & 0.12 \end{bmatrix}$$

$$\mathbf{A}\mathbf{A}^\dagger = \mathbf{I}$$

CHECK

$$LHS = \frac{1}{125}\begin{bmatrix} 1.2 & 0.9 & -4 \\ 1.6 & 1.2 & 1 \end{bmatrix}\begin{bmatrix} 24 & 32 \\ 18 & 24 \\ -24 & 15 \end{bmatrix} = \frac{1}{125}\begin{bmatrix} 125 & 0 \\ 0 & 125 \end{bmatrix} = \mathbf{I} = RHS$$

(c) Since $\boldsymbol{A}$ is of full rank 2 and there are more columns than rows

$$\mathbf{A}^\dagger = \mathbf{A}^T(\mathbf{A}\mathbf{A}^T)^{-1} = \begin{bmatrix} 1.2 & 1.6 \\ 0.9 & 1.2 \\ -4 & 3 \end{bmatrix}\begin{bmatrix} 18.25 & -9 \\ -9 & 13 \end{bmatrix}^{-1} = \frac{1}{156.25}\begin{bmatrix} 1.2 & 1.6 \\ 0.9 & 1.2 \\ -4 & 3 \end{bmatrix}\begin{bmatrix} 13 & 9 \\ 9 & 18.25 \end{bmatrix}$$

$$= \frac{1}{156.25}\begin{bmatrix} 30 & 40 \\ 22.5 & 30 \\ -25 & 18.25 \end{bmatrix} = \begin{bmatrix} 0.192 & 0.256 \\ 0.144 & 0.192 \\ -0.16 & 0.12 \end{bmatrix}$$

which checks with the answer in (b).

▪ **16** (a) Using partitioned matrix multiplication the SVD form of $\boldsymbol{A}$ may be expressed in the
form

$$\mathbf{A} = \hat{\mathbf{U}}\mathbf{\Sigma}\hat{\mathbf{V}}^T = [\,\hat{\mathbf{U}}_r \quad \hat{\mathbf{U}}_{m-r}\,]\begin{bmatrix} \mathbf{S} & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} \hat{\mathbf{V}}_r^T \\ \hat{\mathbf{V}}_{n-r}^T \end{bmatrix} = \hat{\mathbf{U}}_r\mathbf{S}\hat{\mathbf{V}}_r^T$$

(b) Since the diagonal elements in $\boldsymbol{S}$ are non-zero the pseudo inverse may be expressed in the form

$$\mathbf{A}^\dagger = \hat{\mathbf{V}}\mathbf{\Sigma}^*\hat{\mathbf{U}}^T = \hat{\mathbf{V}}_r\mathbf{S}^{-1}\hat{\mathbf{U}}_r^T$$

(c) From the solution to Q46, exercises 1.8.4, the matrix $\mathbf{A} = \begin{bmatrix} 1 & -1 \\ -2 & 2 \\ 2 & -2 \end{bmatrix}$ has a single singularity $\sigma_1 = \sqrt{18}$ so $r = 1$ and $\mathbf{S}$ is a scalar $\sqrt{18}$; $\hat{\mathbf{U}}_r = \hat{\mathbf{U}}_1 = \hat{\mathbf{u}}_1 = [\,\frac{1}{3} \; -\frac{2}{3} \; \frac{2}{3}\,]^T$
and

$$\hat{\mathbf{V}}_r = \hat{\mathbf{V}}_1 = \hat{\mathbf{v}}_1 = [\,\tfrac{1}{\sqrt{2}} \quad -\tfrac{1}{\sqrt{2}}\,]^T$$

The SVD form of $\boldsymbol{A}$ is

$$\mathbf{A} = \hat{\mathbf{u}}_1 S \hat{\mathbf{v}}_1^T = \begin{bmatrix} \frac{1}{3} \\ -\frac{2}{3} \\ \frac{2}{3} \end{bmatrix} \sqrt{18} \left[\frac{1}{\sqrt{2}} \quad -\frac{1}{\sqrt{2}} \right]$$

with direct multiplication confirming $\mathbf{A} = \begin{bmatrix} 1 & -1 \\ -2 & 2 \\ 2 & -2 \end{bmatrix}$

Thus, the pseudo inverse is

$$\mathbf{A}^\dagger = \hat{\mathbf{v}}_1 S^{-1} \hat{\mathbf{u}}_1^T = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix} \frac{1}{\sqrt{18}} \left[\frac{1}{3} \quad -\frac{2}{3} \quad \frac{2}{3} \right] = \begin{bmatrix} \frac{1}{6} \\ -\frac{1}{6} \end{bmatrix} \left[\frac{1}{3} \quad -\frac{2}{3} \quad \frac{2}{3} \right]$$
$$= \frac{1}{18} \begin{bmatrix} 1 & -2 & 2 \\ -1 & 2 & -2 \end{bmatrix}$$

which agrees with the answer obtained in Q46, Exercises 1.8.4

■ **17** $\dot{\mathbf{x}} = \mathbf{A}\ \mathbf{x} + \mathbf{b}u\ ,\ y = \mathbf{c}^T\mathbf{x}$

Let $\lambda_i, e_i, i = 1, 2, \ldots, n,$ be the eigenvalues and corresponding eigenvectors of $\mathbf{A}$. Let $\mathbf{M} = [\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n]$ then since λ_i's are distinct the $\mathbf{e}_i$'s are linearly independent and $\mathbf{M}^{-1}$ exists. Substituting $\mathbf{x} = \mathbf{M}\ \boldsymbol{\xi}$ gives

$$\mathbf{M}\ \dot{\boldsymbol{\xi}} = \mathbf{A}\ \mathbf{M}\ \boldsymbol{\xi} + \mathbf{b}u$$

Premultiplying by M^{-1} gives

$$\dot{\boldsymbol{\xi}} = \mathbf{M}^{-1}\ \mathbf{A}\ \mathbf{M}\ \boldsymbol{\xi} + \mathbf{M}^{-1}\ \mathbf{b}u = \boldsymbol{\Lambda}\ \boldsymbol{\xi} + \mathbf{b}_1 u$$

where $\boldsymbol{\Lambda} = \mathbf{M}^{-1}\ \mathbf{A}\ \mathbf{M} = (\lambda_i \delta_{ij}), i, j = 1, 2, \ldots, n,$ and $\mathbf{b}_1 = \mathbf{M}^{-1}\mathbf{b}$

Also, $y = \mathbf{c}^T\mathbf{x} \ \Rightarrow\ \mathbf{y} = \mathbf{c}^T\mathbf{M}\boldsymbol{\xi} = \mathbf{c}_1^T\boldsymbol{\xi},\ \mathbf{c}_1^T = \mathbf{c}^T\mathbf{M}$. Thus, we have the desired canonical form.

If the vector $\mathbf{b}_1$ contains a zero element then the corresponding mode is uncontrollable and consequently $(\mathbf{A}_1\ \mathbf{b}_1\ \mathbf{c})$ is uncontrollable. If the matrix $\mathbf{c}^T$ has a zero element then the system is unobservable.

The eigenvalues of $\mathbf{A}$ are $\lambda_1 = 2, \lambda_2 = 1, \lambda_3 = -1$ having corresponding eigenvectors $\mathbf{e}_1 = [1\ 3\ 1]^T$, $\mathbf{e}_2 = [3\ 2\ 1]^T$ and $\mathbf{e}_3 = [1\ 0\ 1]^T$.

The modal matrix

$$\mathbf{M} = [\mathbf{e}_1\ \mathbf{e}_2\ \mathbf{e}_3] = \begin{bmatrix} 1 & 3 & 1 \\ 3 & 2 & 0 \\ 1 & 1 & 1 \end{bmatrix} \text{ with } \mathbf{M}^{-1} = -\frac{1}{6}\begin{bmatrix} 2 & -2 & -2 \\ -3 & 0 & 3 \\ 1 & 2 & -7 \end{bmatrix}$$

so canonical form is

$$\begin{bmatrix} \dot{\xi}_1 \\ \dot{\xi}_2 \\ \dot{\xi}_3 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}\begin{bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{bmatrix} + \begin{bmatrix} \frac{1}{3} \\ 0 \\ -\frac{4}{3} \end{bmatrix} u$$

$$y = [1\ \ -4\ \ -2][\xi_1\ \xi_2\ \xi_3]^T$$

We observe that the system is uncontrollable but observable. Since the system matrix $\mathbf{A}$ has positive eigenvalues the system is unstable. Using Kelman matrices

(i) $$\mathbf{A}^2 = \begin{bmatrix} 0 & 1 & 1 \\ -3 & 4 & 3 \\ -1 & 1 & 2 \end{bmatrix}, \quad \mathbf{A}\ \mathbf{b} = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}, \quad \mathbf{A}^2\ \mathbf{b} = \begin{bmatrix} 0 \\ 4 \\ 0 \end{bmatrix}$$

Thus, $[\mathbf{b}\ \ \mathbf{A}\ \mathbf{b}\ \ \mathbf{A}^2\ \mathbf{b}] = \begin{bmatrix} -1 & 2 & 0 \\ 1 & 2 & 4 \\ -1 & 2 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ and is of rank 2

so the system is uncontrollable.

(ii) $[\mathbf{c}\ \ \mathbf{A}^T\ \mathbf{c}\ \ (A^T)^2\mathbf{c}] = \begin{bmatrix} -2 & -3 & -3 \\ 1 & 0 & 2 \\ 0 & 5 & 1 \end{bmatrix} \sim \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ and is of full rank 3

so the system is observable.

■ **18** Model is of form $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$ and making the transformation $\mathbf{x} = \mathbf{M}\mathbf{z}$ gives

$$\mathbf{M}\dot{\mathbf{z}} = \mathbf{A}\mathbf{M}\mathbf{z} + \mathbf{B}\mathbf{u} \Rightarrow \dot{\mathbf{z}} = \mathbf{M}^{-1}\mathbf{A}\mathbf{M}\mathbf{z} + \mathbf{M}^{-1}\mathbf{B}\mathbf{u} \Rightarrow \dot{\mathbf{z}} = \mathbf{\Lambda}\mathbf{z} + \mathbf{M}^{-1}\mathbf{B}\mathbf{u}$$

where $\mathbf{M}$ and $\mathbf{\Lambda}$ are respectively the modal and spectral matrices of $\mathbf{A}$.
The eigenvalues of $\boldsymbol{A}$ are given by

$$\begin{vmatrix} -2-\lambda & -2 & 0 \\ 0 & -\lambda & 1 \\ 0 & -3 & -4-\lambda \end{vmatrix} = 0 \Rightarrow -(2-\lambda)(4\lambda + \lambda^2 + 3) = 0$$

$$\Rightarrow (\lambda + 2)(\lambda + 1)(\lambda + 3) = 0$$

$$\Rightarrow \lambda_1 - 1, \lambda_2 = -2, \lambda_3 = -3$$

with corresponding eigenvectors

$$\mathbf{e}_1 = [-2 \quad 1 \quad -2]^T, \mathbf{e}_2 = [1 \quad 0 \quad 0]^T \text{ and } \mathbf{e}_3 = [-2 \quad -1 \quad 3]^T$$

Thus, the modal and spectral matrices are

$$\mathbf{M} = \begin{bmatrix} -2 & 1 & -2 \\ 1 & 0 & -4 \\ -1 & 0 & -1 \end{bmatrix} \text{ and} \mathbf{\Lambda} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -3 \end{bmatrix}$$

and $\det \mathbf{M} = -2 \Rightarrow \mathbf{M}^{-1} = \begin{bmatrix} 0 & \frac{3}{2} & \frac{1}{2} \\ 1 & 4 & 2 \\ 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix} \Rightarrow \mathbf{M}^{-1}\mathbf{B} = \begin{bmatrix} 0 & \frac{3}{2} & \frac{1}{2} \\ 1 & 4 & 2 \\ 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}$

$= \begin{bmatrix} \frac{1}{2} & 2 \\ 3 & 6 \\ \frac{1}{2} & 1 \end{bmatrix}$ leading to the canonical form

$$\dot{\mathbf{z}} = \begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \\ \dot{z}_3 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -3 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} + \begin{bmatrix} \frac{1}{2} & 2 \\ 3 & 6 \\ \frac{1}{2} & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

From (1.99a) the solution is given by

$$\begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} e^{-t} & 0 & 0 \\ 0 & e^{-2t} & 0 \\ 0 & 0 & e^{-3t} \end{bmatrix} \begin{bmatrix} z_1(0) \\ z_2(0) \\ z_3(0) \end{bmatrix} + \int_0^t \begin{bmatrix} e^{-(t-\tau)} & 0 & 0 \\ 0 & e^{-2(t-\tau)} & 0 \\ 0 & 0 & e^{-3(t-\tau)} \end{bmatrix}$$

$$\begin{bmatrix} \frac{1}{2} & 2 \\ 3 & 6 \\ \frac{1}{2} & 1 \end{bmatrix} \begin{bmatrix} \tau \\ 1 \end{bmatrix} d\tau$$

with $\mathbf{z}(0) = \mathbf{M}^{-1}\mathbf{x}(0) = \begin{bmatrix} 0 & \frac{3}{2} & \frac{1}{2} \\ 1 & 4 & 2 \\ 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 10 \\ 5 \\ 2 \end{bmatrix} = [\frac{17}{2} \quad 34 \quad \frac{7}{2}]^T$. Thus,

$$\mathbf{z} = \begin{bmatrix} \frac{17}{2}e^t \\ 34e^{-2t} \\ \frac{7}{2}e^{-3t} \end{bmatrix} + \int_0^t \begin{bmatrix} (2+\frac{1}{2}\tau)e^{-(t-\tau)} \\ (6+3\tau)e^{-2(t-\tau)} \\ (1+\frac{1}{2}\tau)e^{-3(t-\tau)} \end{bmatrix} d\tau \Rightarrow \mathbf{z} = \begin{bmatrix} \frac{17}{2}e^t \\ 34e^{-2t} \\ \frac{7}{2}e^{-3t} \end{bmatrix}$$

$$+ \begin{bmatrix} \frac{1}{2}t + \frac{3}{2} - \frac{3}{2}e^{-t} \\ \frac{3}{2}t + \frac{9}{4} - \frac{9}{4}e^{-2t} \\ \frac{1}{6}t + \frac{5}{18} - \frac{5}{18}e^{-3t} \end{bmatrix} \Rightarrow \mathbf{z} = \begin{bmatrix} \frac{1}{2}t + \frac{3}{2} + 7e^{-t} \\ \frac{3}{2}t + \frac{9}{4} - \frac{127}{4}e^{-2t} \\ \frac{1}{6}t + \frac{5}{18} + \frac{29}{9}e^{-3t} \end{bmatrix}$$

giving $\mathbf{x} = \mathbf{Mz} = \begin{bmatrix} -2 & 1 & -2 \\ 1 & 0 & -1 \\ -1 & 0 & 3 \end{bmatrix} \begin{bmatrix} \frac{1}{2}t + \frac{3}{2} + 7e^{-t} \\ \frac{3}{2}t + \frac{9}{4} - \frac{127}{4}e^{-2t} \\ \frac{1}{6}t + \frac{5}{18} + \frac{29}{9}e^{-3t} \end{bmatrix}$

$$\Rightarrow \mathbf{x}(t) = \begin{bmatrix} -14e^{-t} + \frac{127}{4}e^{-2t} - \frac{58}{9}e^{-3t} + \frac{1}{6}t - \frac{47}{36} \\ 7e^{-t} - \frac{29}{9}e^{-3t} + \frac{1}{3}t + \frac{11}{9} \\ -7e^{-t} + \frac{29}{3}e^{-3t} - \frac{2}{3} \end{bmatrix}$$

- **19(a)** Eigenvalues of the matrix given by

$$0 = \begin{vmatrix} 5-\lambda & 2 & -1 \\ 3 & 6-\lambda & -9 \\ 1 & 1 & 1-\lambda \end{vmatrix} \overset{C_1 - C_2}{=} \begin{vmatrix} 3-\lambda & 2 & -1 \\ -3+\lambda & 6-\lambda & -9 \\ 0 & 1 & 1-\lambda \end{vmatrix}$$

$$= (3-\lambda) \begin{vmatrix} 1 & 2 & -1 \\ 0 & 8-\lambda & -10 \\ 0 & 1 & 1-\lambda \end{vmatrix}$$

$$= (3-\lambda)(\lambda^2 - 9\lambda + 18) = (3-\lambda)(\lambda-3)(\lambda-6)$$

so the eigenvalues are $\lambda_1 = 6, \lambda_2 = \lambda_3 = 3$

When $\lambda = 3, \mathbf{A} - 3\mathbf{I} = \begin{bmatrix} 2 & 2 & -1 \\ 3 & 3 & -9 \\ 1 & 1 & -2 \end{bmatrix} \sim \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ is of rank 2

so there is only $3 - 2 = 1$ corresponding eigenvectors.

The eigenvector corresponding to $\lambda_1 = 6$ is readily determined as $\mathbf{e}_1 = [3\ 2\ 1]^T$.

Likewise the single eigenvector corresponding to $\lambda_2 = 6$ is determined as

$$\mathbf{e}_2 = [1\ \ -1\ 0]^T$$

The generalized eigenvector $\mathbf{e}_2^*$ determined by

$$(\mathbf{A} - 2\mathbf{I})\mathbf{e}_2^* = \mathbf{e}_2$$

$$\text{or} \quad 3e_{21}^* + 2e_{22}^* - e_{23}^* = 1$$

$$3e_{21}^* + 3e_{22}^* - 9e_{23}^* = -1$$

$$e_{21}^* + e_{22}^* - 2e_{23}^* = 0$$

giving $\mathbf{e}_2^* = [\ \frac{1}{3}\ \frac{1}{3}\ \frac{1}{3}]^T$.

For convenience, we can take the two eigenvectors corresponding to $\lambda = 3$ as

$$\mathbf{e}_2 = [3\ \ -3\ 0]^T,\ \mathbf{e}_2^* = [1\ 1\ 1]^T$$

The corresponding Jordan canonical form being $\mathbf{J} = \begin{bmatrix} 6 & 0 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 3 \end{bmatrix}$

19(b) The generalised modal matrix is then

$$\mathbf{M} = \begin{bmatrix} 3 & -3 & 1 \\ 2 & -3 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

$$\mathbf{A}\ \mathbf{M} = \begin{bmatrix} 5 & 2 & -1 \\ 3 & 6 & -9 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 3 & -3 & 1 \\ 2 & -3 & 1 \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 18 & 9 & 6 \\ 12 & -9 & 0 \\ 6 & 0 & 3 \end{bmatrix}$$

$$\mathbf{M}\ \mathbf{J} = \begin{bmatrix} 3 & 3 & 1 \\ 2 & -3 & 1 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 6 & 0 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 13 & 9 & 6 \\ 12 & -9 & 0 \\ 6 & 0 & 3 \end{bmatrix}$$

so $\mathbf{A}\ \mathbf{M} = \mathbf{M}\ \mathbf{J}$

19(c) $\mathbf{M}^{-1} = -\dfrac{1}{9} \begin{bmatrix} -3 & -3 & 6 \\ -1 & 2 & -1 \\ 3 & 3 & -15 \end{bmatrix}, \ e^{\mathbf{J}t} = \begin{bmatrix} e^{6t} & 0 & 0 \\ 0 & e^{3t} & te^{3t} \\ 0 & 0 & e^{3t} \end{bmatrix}$

so

$$\mathbf{x}(t) = -\frac{1}{9} \begin{bmatrix} 3 & 3 & 1 \\ 2 & -3 & 1 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} e^{6t} & 0 & 0 \\ 0 & e^{3t} & te^{3t} \\ 0 & 0 & e^{3t} \end{bmatrix} \begin{bmatrix} -3 & -3 & 6 \\ -1 & 2 & -1 \\ 3 & 3 & -15 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$
$$= \frac{1}{9} \begin{bmatrix} 9e^{6t} - 9(1+t)e^{3t} \\ 6e^{6t} + (3+9t)e^{3t} \\ 3e^{6t} - 3e^{3t} \end{bmatrix}$$

■ **20** Substituting $\mathbf{x} = e^{\lambda t}\mathbf{u}$, where $\mathbf{u}$ is a constant vector, in $\mathbf{x} = \mathbf{A}\ \mathbf{x}$ gives

$$\lambda^2\mathbf{u} = \mathbf{A}\ \mathbf{u} \text{ or } (\mathbf{A} - \lambda^2\mathbf{I})\mathbf{u} = 0 \qquad (1)$$

so that there is a non-trivial solution provided

$$|\ \mathbf{A} - \lambda^2\mathbf{I}\ | = 0 \qquad (2)$$

If $\lambda_1^2, \lambda_2^2, \ldots, \lambda_n^2$ are the solutions of (2) and $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n$ the corresponding solutions of (1) define

$$\mathbf{M} = [\mathbf{u}_1 \ \mathbf{u}_2 \ \ldots \ \mathbf{u}_n] \text{ and } \mathbf{S} = \text{diag } (\lambda_1^2 \ \lambda_2^2 \ \ldots \ \lambda_n^2)$$

Applying the transformation $\mathbf{x} = \mathbf{M}\ \mathbf{q}$, $\mathbf{q} = [q_1 \ q_2 \ \ldots \ q_n]$ gives

$$\mathbf{M}\ \ddot{\mathbf{q}} = \mathbf{A}\ \mathbf{M}\ \mathbf{q}$$

giving $\ddot{\mathbf{q}} = \mathbf{M}^{-1}\ \mathbf{A}\ \mathbf{M}\ \mathbf{q}$ provided $\mathbf{u}_1,\ \mathbf{u}_2, \ldots, u_n$ are linearly independent

so that $\ddot{\mathbf{q}} = \mathbf{S}\ \mathbf{q}$ since $\mathbf{M}^{-1}\ \mathbf{A}\ \mathbf{M} = \mathbf{S}$

This represents n differential equations of the form

$$\ddot{q}_i = \lambda_i^2 q_i \ , \ i = 1, 2, \ldots, n$$

When $\lambda_i^2 < 0$ this has the solution of the form

$$q_i = C_i \sin(\omega_i t + \alpha_i)$$

where C_i and α_i are arbitrary constants and $\lambda_i = j\omega_i$
The given differential equations may be written in the vector–matrix form

$$\dot{\mathbf{x}} = \begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} = \begin{bmatrix} -3 & 2 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

which is of the above form

$$\ddot{\mathbf{x}} = \mathbf{A}\ \mathbf{x}$$

$0 = \mid \mathbf{A} - \lambda^2 \mathbf{I} \mid$ gives $(\lambda^2)^2 + 5(\lambda^2) + 4 = 0$ or $\lambda_1^2 = -1, \lambda_2^2 = -4$.
Solving the corresponding equation

$$(\mathbf{A} - \lambda_i^2 \mathbf{I})\ \mathbf{u}_i = 0$$

we have that $\mathbf{u}_1 = [1\ 1]^T$ and $\mathbf{u}_2 = [2\ \ -1]^T$. Thus, we take

$$\mathbf{M} = \begin{bmatrix} 1 & 2 \\ 1 & -1 \end{bmatrix} \quad \text{and} \quad \mathbf{S} = \begin{bmatrix} -1 & 0 \\ 0 & -4 \end{bmatrix}$$

The normal modes of the system are given by

$$\begin{bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -4 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \end{bmatrix}$$

giving

$$q_1(t) = C_1 \sin(t + \alpha_1) \equiv \gamma_1 \sin t + \beta_1 \cos t$$

$$q_2(t) = C_2 \sin(2t + \alpha_2) \equiv \gamma_2 \sin 2t + \beta_2 \cos 2t$$

Since $\mathbf{x} = \mathbf{M}\ \mathbf{q}$ we have that $\mathbf{q}(0) = \mathbf{M}^{-1}\mathbf{x}(0) = -\dfrac{1}{3}\begin{bmatrix} -1 & -2 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} \frac{5}{3} \\ -\frac{1}{3} \end{bmatrix}$
also $\dot{\mathbf{q}}(0) = \mathbf{M}^{-1}\dot{\mathbf{x}}(0)$ so that $\dot{q}_1(0) = 2$ and $\dot{q}_2(0) = 0$
Using these initial conditions we can determine $\gamma_1, \beta_1, \gamma_2$ and β_2 to give

$$q_1(t) = \frac{5}{3}\cos t + 2\sin t$$

$$q_2(t) = -\frac{1}{3}\cos 2t$$

The general displacements $x_1(t)$ and $x_2(t)$ are then given by $\mathbf{x} = \mathbf{M}\ \mathbf{q}$ so

$$x_1 = q_1 + 2q_2 = \frac{5}{3}\cos t + 2\sin t - \frac{2}{3}\cos 2t$$

$$x_2 = q_1 - q_2 = \frac{5}{3}\cos t + 2\sin t - \frac{1}{3}\cos 2t$$
