CONTENTS

NOTE: "Solutions for Chapter 18-cosv9" is only needed by users of *Calculus of Several Variables* (9th Edition), which includes extra material in Sections 18.2 and 18.5 that is found in *Calculus: a Complete Course* and in *Single-Variable Calculus* in Sections 7.9 and 3.7 respectively. Solutions for Chapter 18-cosv9 contains only the solutions for the two Sections 18.2 and 18.9 in the Several Variables book. All other Sections are in "Solutions for Chapter 18."

It should also be noted that some of the material in Chapter 18 is beyond the scope of most students in single-variable calculus courses as it requires the use of multivariable functions and partial derivatives.

CHAPTER P. PRELIMINARIES

Section P.1 Real Numbers and the Real Line (page 10)

1. $\frac{2}{9} = 0.22222222\cdots = 0.\overline{2}$

2. $\frac{1}{11} = 0.09090909\cdots = 0.\overline{09}$

3. If $x = 0.121212\cdots$, then $100x = 12.121212\cdots = 12 + x$. Thus $99x = 12$ and $x = 12/99 = 4/33$.

4. If $x = 3.277777\cdots$, then $10x - 32 = 0.77777\cdots$ and $100x - 320 = 7 + (10x - 32)$, or $90x = 295$. Thus $x = 295/90 = 59/18$.

5. $1/7 = 0.142857142857\cdots = 0.\overline{142857}$
$2/7 = 0.285714285714\cdots = 0.\overline{285714}$
$3/7 = 0.428571428571\cdots = 0.\overline{428571}$
$4/7 = 0.571428571428\cdots = 0.\overline{571428}$
note the same cyclic order of the repeating digits
$5/7 = 0.714285714285\cdots = 0.\overline{714285}$
$6/7 = 0.857142857142\cdots = 0.\overline{857142}$

6. Two different decimal expansions can represent the same number. For instance, both $0.999999\cdots = 0.\overline{9}$ and $1.000000\cdots = 1.\overline{0}$ represent the number 1.

7. $x \ge 0$ and $x \le 5$ define the interval $[0, 5]$.

8. $x < 2$ and $x \ge -3$ define the interval $[-3, 2)$.

9. $x > -5$ or $x < -6$ defines the union $(-\infty, -6) \cup (-5, \infty)$.

10. $x \le -1$ defines the interval $(-\infty, -1]$.

11. $x > -2$ defines the interval $(-2, \infty)$.

12. $x < 4$ or $x \ge 2$ defines the interval $(-\infty, \infty)$, that is, the whole real line.

13. If $-2x > 4$, then $x < -2$. Solution: $(-\infty, -2)$

14. If $3x + 5 \le 8$, then $3x \le 8 - 5 - 3$ and $x \le 1$. Solution: $(-\infty, 1]$

15. If $5x - 3 \le 7 - 3x$, then $8x \le 10$ and $x \le 5/4$. Solution: $(-\infty, 5/4]$

16. If $\frac{6-x}{4} \ge \frac{3x-4}{2}$, then $6 - x \ge 6x - 8$. Thus $14 \ge 7x$ and $x \le 2$. Solution: $(-\infty, 2]$

17. If $3(2 - x) < 2(3 + x)$, then $0 < 5x$ and $x > 0$. Solution: $(0, \infty)$

18. If $x^2 < 9$, then $|x| < 3$ and $-3 < x < 3$. Solution: $(-3, 3)$

19. Given: $1/(2 - x) < 3$.
CASE I. If $x < 2$, then $1 < 3(2 - x) = 6 - 3x$, so $3x < 5$ and $x < 5/3$. This case has solutions $x < 5/3$.
CASE II. If $x > 2$, then $1 > 3(2 - x) = 6 - 3x$, so $3x > 5$ and $x > 5/3$. This case has solutions $x > 2$.
Solution: $(-\infty, 5/3) \cup (2, \infty)$.

20. Given: $(x + 1)/x \ge 2$.
CASE I. If $x > 0$, then $x + 1 \ge 2x$, so $x \le 1$.
CASE II. If $x < 0$, then $x + 1 \le 2x$, so $x \ge 1$. (not possible)
Solution: $(0, 1]$.

21. Given: $x^2 - 2x \le 0$. Then $x(x - 2) \le 0$. This is only possible if $x \ge 0$ and $x \le 2$. Solution: $[0, 2]$.

22. Given $6x^2 - 5x \le -1$, then $(2x - 1)(3x - 1) \le 0$, so either $x \le 1/2$ and $x \ge 1/3$, or $x \le 1/3$ and $x \ge 1/2$. The latter combination is not possible. The solution set is $[1/3, 1/2]$.

23. Given $x^3 > 4x$, we have $x(x^2 - 4) > 0$. This is possible if $x < 0$ and $x^2 < 4$, or if $x > 0$ and $x^2 > 4$. The possibilities are, therefore, $-2 < x < 0$ or $2 < x < \infty$. Solution: $(-2, 0) \cup (2, \infty)$.

24. Given $x^2 - x \le 2$, then $x^2 - x - 2 \le 0$ so $(x - 2)(x + 1) \le 0$. This is possible if $x \le 2$ and $x \ge -1$ or if $x \ge 2$ and $x \le -1$. The latter situation is not possible. The solution set is $[-1, 2]$.

25. Given: $\frac{x}{2} \ge 1 + \frac{4}{x}$.
CASE I. If $x > 0$, then $x^2 \ge 2x + 8$, so that $x^2 - 2x - 8 \ge 0$, or $(x - 4)(x + 2) \ge 0$. This is possible for $x > 0$ only if $x \ge 4$.
CASE II. If $x < 0$, then we must have $(x - 4)(x + 2) \le 0$, which is possible for $x < 0$ only if $x \ge -2$.
Solution: $[-2, 0) \cup [4, \infty)$.

26. Given: $\frac{3}{x-1} < \frac{2}{x+1}$.
CASE I. If $x > 1$ then $(x - 1)(x + 1) > 0$, so that $3(x + 1) < 2(x - 1)$. Thus $x < -5$. There are no solutions in this case.
CASE II. If $-1 < x < 1$, then $(x - 1)(x + 1) < 0$, so $3(x + 1) > 2(x - 1)$. Thus $x > -5$. In this case all numbers in $(-1, 1)$ are solutions.
CASE III. If $x < -1$, then $(x - 1)(x + 1) > 0$, so that $3(x + 1) < 2(x - 1)$. Thus $x < -5$. All numbers $x < -5$ are solutions.
Solutions: $(-\infty, -5) \cup (-1, 1)$.

27. If $|x| = 3$ then $x = \pm 3$.

28. If $|x - 3| = 7$, then $x - 3 = \pm 7$, so $x = -4$ or $x = 10$.

29. If $|2t + 5| = 4$, then $2t + 5 = \pm 4$, so $t = -9/2$ or $t = -1/2$.

30. If $|1 - t| = 1$, then $1 - t = \pm 1$, so $t = 0$ or $t = 2$.

31. If $|8 - 3s| = 9$, then $8 - 3s = \pm 9$, so $3s = -1$ or 17, and $s = -1/3$ or $s = 17/3$.

32. If $\left|\frac{s}{2}-1\right| = 1$, then $\frac{s}{2} - 1 = \pm 1$, so $s = 0$ or $s = 4$.

33. If $|x| < 2$, then x is in $(-2, 2)$.

34. If $|x| \le 2$, then x is in $[-2, 2]$.

35. If $|s - 1| \le 2$, then $1 - 2 \le s \le 1 + 2$, so s is in $[-1, 3]$.

36. If $|t + 2| < 1$, then $-2 - 1 < t < -2 + 1$, so t is in $(-3, -1)$.

37. If $|3x - 7| < 2$, then $7 - 2 < 3x < 7 + 2$, so x is in $(5/3, 3)$.

38. If $|2x + 5| < 1$, then $-5 - 1 < 2x < -5 + 1$, so x is in $(-3, -2)$.

39. If $\left|\frac{x}{2}-1\right| \le 1$, then $1 - 1 \le \frac{x}{2} \le 1 + 1$, so x is in $[0, 4]$.

40. If $\left|2 - \frac{x}{2}\right| < \frac{1}{2}$, then $x/2$ lies between $2 - (1/2)$ and $2 + (1/2)$. Thus x is in $(3, 5)$.

41. The inequality $|x + 1| > |x - 3|$ says that the distance from x to -1 is greater than the distance from x to 3, so x must be to the right of the point half-way between -1 and 3. Thus $x > 1$.

42. $|x - 3| < 2|x| \Leftrightarrow x^2 - 6x + 9 = (x - 3)^2 < 4x^2$ $\Leftrightarrow 3x^2 + 6x - 9 > 0 \Leftrightarrow 3(x + 3)(x - 1) > 0$. This inequality holds if $x < -3$ or $x > 1$.

43. $|a| = a$ if and only if $a \ge 0$. It is false if $a < 0$.

44. The equation $|x - 1| = 1 - x$ holds if $|x - 1| = -(x - 1)$, that is, if $x - 1 \le 0$, or, equivalently, if $x \le 1$.

45. The triangle inequality $|x + y| \le |x| + |y|$ implies that

$$|x| \ge |x + y| - |y|.$$

Apply this inequality with $x = a - b$ and $y = b$ to get

$$|a - b| \ge |a| - |b|.$$

Similarly, $|a - b| = |b - a| \ge |b| - |a|$. Since $\big||a| - |b|\big|$ is equal to either $|a| - |b|$ or $|b| - |a|$, depending on the sizes of a and b, we have

$$|a - b| \ge \big||a| - |b|\big|.$$

Section P.2 Cartesian Coordinates in the Plane (page 16)

1. From $A(0, 3)$ to $B(4, 0)$, $\Delta x = 4 - 0 = 4$ and $\Delta y = 0 - 3 = -3$. $|AB| = \sqrt{4^2 + (-3)^2} = 5$.

2. From $A(-1, 2)$ to $B(4, -10)$, $\Delta x = 4 - (-1) = 5$ and $\Delta y = -10 - 2 = -12$. $|AB| = \sqrt{5^2 + (-12)^2} = 13$.

3. From $A(3, 2)$ to $B(-1, -2)$, $\Delta x = -1 - 3 = -4$ and $\Delta y = -2 - 2 = -4$. $|AB| = \sqrt{(-4)^2 + (-4)^2} = 4\sqrt{2}$.

4. From $A(0.5, 3)$ to $B(2, 3)$, $\Delta x = 2 - 0.5 = 1.5$ and $\Delta y = 3 - 3 = 0$. $|AB| = 1.5$.

5. Starting point: $(-2, 3)$. Increments $\Delta x = 4$, $\Delta y = -7$. New position is $(-2 + 4, 3 + (-7))$, that is, $(2, -4)$.

6. Arrival point: $(-2, -2)$. Increments $\Delta x = -5$, $\Delta y = 1$. Starting point was $(-2 - (-5), -2 - 1)$, that is, $(3, -3)$.

7. $x^2 + y^2 = 1$ represents a circle of radius 1 centred at the origin.

8. $x^2 + y^2 = 2$ represents a circle of radius $\sqrt{2}$ centred at the origin.

9. $x^2 + y^2 \le 1$ represents points inside and on the circle of radius 1 centred at the origin.

10. $x^2 + y^2 = 0$ represents the origin.

11. $y \ge x^2$ represents all points lying on or above the parabola $y = x^2$.

12. $y < x^2$ represents all points lying below the parabola $y = x^2$.

13. The vertical line through $(-2, 5/3)$ is $x = -2$; the horizontal line through that point is $y = 5/3$.

14. The vertical line through $(\sqrt{2}, -1.3)$ is $x = \sqrt{2}$; the horizontal line through that point is $y = -1.3$.

15. Line through $(-1, 1)$ with slope $m = 1$ is $y = 1 + 1(x + 1)$, or $y = x + 2$.

16. Line through $(-2, 2)$ with slope $m = 1/2$ is $y = 2 + (1/2)(x + 2)$, or $x - 2y = -6$.

17. Line through $(0, b)$ with slope $m = 2$ is $y = b + 2x$.

18. Line through $(a, 0)$ with slope $m = -2$ is $y = 0 - 2(x - a)$, or $y = 2a - 2x$.

19. At $x = 2$, the height of the line $2x + 3y = 6$ is $y = (6 - 4)/3 = 2/3$. Thus $(2, 1)$ lies above the line.

20. At $x = 3$, the height of the line $x - 4y = 7$ is $y = (3 - 7)/4 = -1$. Thus $(3, -1)$ lies on the line.

21. The line through $(0, 0)$ and $(2, 3)$ has slope $m = (3 - 0)/(2 - 0) = 3/2$ and equation $y = (3/2)x$ or $3x - 2y = 0$.

22. The line through $(-2, 1)$ and $(2, -2)$ has slope $m = (-2 - 1)/(2 + 2) = -3/4$ and equation $y = 1 - (3/4)(x + 2)$ or $3x + 4y = -2$.

23. The line through $(4, 1)$ and $(-2, 3)$ has slope $m = (3 - 1)/(-2 - 4) = -1/3$ and equation $y = 1 - \frac{1}{3}(x - 4)$ or $x + 3y = 7$.

24. The line through $(-2, 0)$ and $(0, 2)$ has slope $m = (2 - 0)/(0 + 2) = 1$ and equation $y = 2 + x$.

25. If $m = -2$ and $b = \sqrt{2}$, then the line has equation $y = -2x + \sqrt{2}$.

26. If $m = -1/2$ and $b = -3$, then the line has equation $y = -(1/2)x - 3$, or $x + 2y = -6$.

27. $3x + 4y = 12$ has x-intercept $a = 12/3 = 4$ and y-intercept $b = 12/4 = 3$. Its slope is $-b/a = -3/4$.

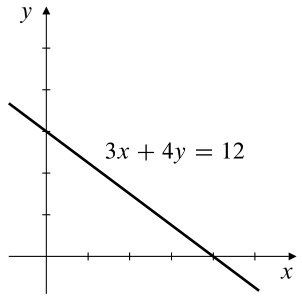


Fig. P.2-27

28. $x + 2y = -4$ has x-intercept $a = -4$ and y-intercept $b = -4/2 = -2$. Its slope is $-b/a = 2/(-4) = -1/2$.

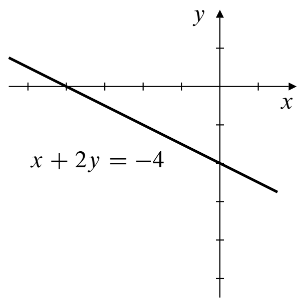


Fig. P.2-28

29. $\sqrt{2}x - \sqrt{3}y = 2$ has x-intercept $a = 2/\sqrt{2} = \sqrt{2}$ and y-intercept $b = -2/\sqrt{3}$. Its slope is $-b/a = 2/\sqrt{6} = \sqrt{2/3}$.

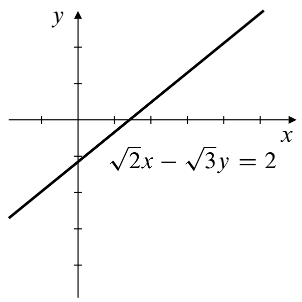


Fig. P.2-29

30. $1.5x - 2y = -3$ has x-intercept $a = -3/1.5 = -2$ and y-intercept $b = -3/(-2) = 3/2$. Its slope is $-b/a = 3/4$.

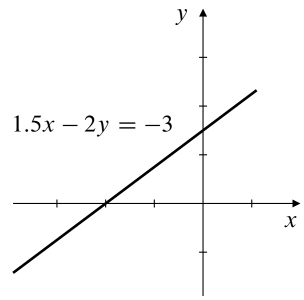


Fig. P.2-30

31. line through $(2, 1)$ parallel to $y = x + 2$ is $y = x - 1$; line perpendicular to $y = x + 2$ is $y = -x + 3$.

32. line through $(-2, 2)$ parallel to $2x + y = 4$ is $2x + y = -2$; line perpendicular to $2x + y = 4$ is $x - 2y = -6$.

33. We have

$$\begin{aligned} 3x + 4y &= -6 \quad \Longrightarrow \quad & 6x + 8y &= -12 \\ 2x - 3y &= 13 & 6x - 9y &= 39. \end{aligned}$$

Subtracting these equations gives $17y = -51$, so $y = -3$ and $x = (13 - 9)/2 = 2$. The intersection point is $(2, -3)$.

34. We have

$$\begin{aligned} 2x + \ \ y &= 8 \quad \Longrightarrow \quad & 14x + 7y &= 56 \\ 5x - 7y &= 1 & 5x - 7y &= 1. \end{aligned}$$

Adding these equations gives $19x = 57$, so $x = 3$ and $y = 8 - 2x = 2$. The intersection point is $(3, 2)$.

35. If $a \neq 0$ and $b \neq 0$, then $(x/a) + (y/b) = 1$ represents a straight line that is neither horizontal nor vertical, and does not pass through the origin. Putting $y = 0$ we get $x/a = 1$, so the x-intercept of this line is $x = a$; putting $x = 0$ gives $y/b = 1$, so the y-intercept is $y = b$.

36. The line $(x/2) - (y/3) = 1$ has x-intercept $a = 2$, and y-intercept $b = -3$.

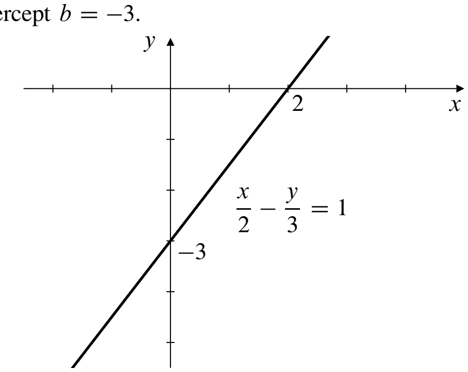


Fig. P.2-36

37. The line through $(2, 1)$ and $(3, -1)$ has slope $m = (-1 - 1)/(3 - 2) = -2$ and equation $y = 1 - 2(x - 2) = 5 - 2x$. Its y-intercept is 5.

38. The line through $(-2, 5)$ and $(k, 1)$ has x-intercept 3, so also passes through $(3, 0)$. Its slope m satisfies

$$\frac{1-0}{k-3} = m = \frac{0-5}{3+2} = -1.$$

Thus $k - 3 = -1$, and so $k = 2$.

39. $C = Ax + B$. If $C = 5,000$ when $x = 10,000$ and $C = 6,000$ when $x = 15,000$, then

$$\begin{aligned} 10,000A + B &= 5,000 \\ 15,000A + B &= 6,000 \end{aligned}$$

Subtracting these equations gives $5,000A = 1,000$, so $A = 1/5$. From the first equation, $2,000 + B = 5,000$, so $B = 3,000$. The cost of printing 100,000 pamphlets is $\$100,000/5 + 3,000 = \$23,000$.

40. $-40°$ and $-40°$ is the same temperature on both the Fahrenheit and Celsius scales.

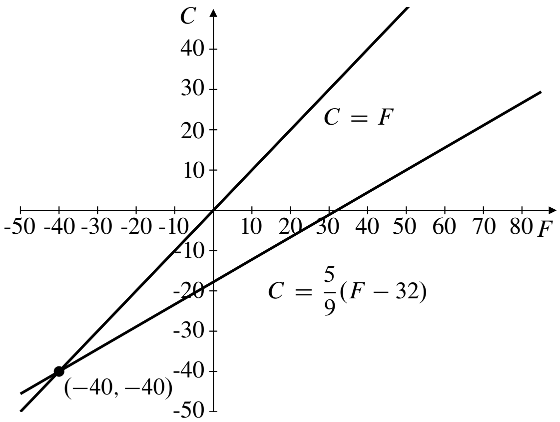


Fig. P.2-40

41. $A = (2, 1), \quad B = (6, 4), \quad C = (5, -3)$

$$\begin{aligned} |AB| &= \sqrt{(6-2)^2 + (4-1)^2} = \sqrt{25} = 5 \\ |AC| &= \sqrt{(5-2)^2 + (-3-1)^2} = \sqrt{25} = 5 \\ |BC| &= \sqrt{(6-5)^2 + (4+3)^2} = \sqrt{50} = 5\sqrt{2}. \end{aligned}$$

Since $|AB| = |AC|$, triangle ABC is isosceles.

42. $A = (0, 0), \quad B = (1, \sqrt{3}), \quad C = (2, 0)$

$$\begin{aligned} |AB| &= \sqrt{(1-0)^2 + (\sqrt{3}-0)^2} = \sqrt{4} = 2 \\ |AC| &= \sqrt{(2-0)^2 + (0-0)^2} = \sqrt{4} = 2 \\ |BC| &= \sqrt{(2-1)^2 + (0-\sqrt{3})^2} = \sqrt{4} = 2. \end{aligned}$$

Since $|AB| = |AC| = |BC|$, triangle ABC is equilateral.

43. $A = (2, -1), \quad B = (1, 3), \quad C = (-3, 2)$

$$\begin{aligned} |AB| &= \sqrt{(1-2)^2 + (3+1)^2} = \sqrt{17} \\ |AC| &= \sqrt{(-3-2)^2 + (2+1)^2} = \sqrt{34} = \sqrt{2}\sqrt{17} \\ |BC| &= \sqrt{(-3-1)^2 + (2-3)^2} = \sqrt{17}. \end{aligned}$$

Since $|AB| = |BC|$ and $|AC| = \sqrt{2}|AB|$, triangle ABC is an isosceles right-angled triangle with right angle at B. Thus $ABCD$ is a square if D is displaced from C by the same amount A is from B, that is, by increments $\Delta x = 2 - 1 = 1$ and $\Delta y = -1 - 3 = -4$. Thus $D = (-3 + 1, 2 + (-4)) = (-2, -2)$.

44. If $M = (x_m, y_m)$ is the midpoint of P_1P_2, then the displacement of M from P_1 equals the displacement of P_2 from M:

$$x_m - x_1 = x_2 - x_m, \quad y_m - y_1 = y_2 - y_m.$$

Thus $x_m = (x_1 + x_2)/2$ and $y_m = (y_1 + y_2)/2$.

45. If $Q = (x_q, y_q)$ is the point on P_1P_2 that is two thirds of the way from P_1 to P_2, then the displacement of Q from P_1 equals twice the displacement of P_2 from Q:

$$x_q - x_1 = 2(x_2 - x_q), \quad y_q - y_1 = 2(y_2 - y_q).$$

Thus $x_q = (x_1 + 2x_2)/3$ and $y_q = (y_1 + 2y_2)/3$.

46. Let the coordinates of P be $(x, 0)$ and those of Q be $(X, -2X)$. If the midpoint of PQ is $(2, 1)$, then

$$(x + X)/2 = 2, \quad (0 - 2X)/2 = 1.$$

The second equation implies that $X = -1$, and the second then implies that $x = 5$. Thus P is $(5, 0)$.

47. $\sqrt{(x-2)^2 + y^2} = 4$ says that the distance of (x, y) from $(2, 0)$ is 4, so the equation represents a circle of radius 4 centred at $(2, 0)$.

48. $\sqrt{(x-2)^2 + y^2} = \sqrt{x^2 + (y-2)^2}$ says that (x, y) is equidistant from $(2, 0)$ and $(0, 2)$. Thus (x, y) must lie on the line that is the right bisector of the line from $(2, 0)$ to $(0, 2)$. A simpler equation for this line is $x = y$.

49. The line $2x + ky = 3$ has slope $m = -2/k$. This line is perpendicular to $4x + y = 1$, which has slope -4, provided $m = 1/4$, that is, provided $k = -8$. The line is parallel to $4x + y = 1$ if $m = -4$, that is, if $k = 1/2$.

50. For any value of k, the coordinates of the point of intersection of $x + 2y = 3$ and $2x - 3y = -1$ will also satisfy the equation

$$(x + 2y - 3) + k(2x - 3y + 1) = 0$$

because they cause both expressions in parentheses to be 0. The equation above is linear in x and y, and so represents a straight line for any choice of k. This line will pass through $(1, 2)$ provided $1 + 4 - 3 + k(2 - 6 + 1) = 0$, that is, if $k = 2/3$. Therefore, the line through the point of intersection of the two given lines and through the point $(1, 2)$ has equation

$$x + 2y - 3 + \frac{2}{3}(2x - 3y + 1) = 0,$$

or, on simplification, $x = 1$.

Section P.3 Graphs of Quadratic Equations (page 22)

1. $x^2 + y^2 = 16$

2. $x^2 + (y - 2)^2 = 4$, or $x^2 + y^2 - 4y = 0$

3. $(x + 2)^2 + y^2 = 9$, or $x^2 + y^2 + 4y = 5$

4. $(x - 3)^2 + (y + 4)^2 = 25$, or $x^2 + y^2 - 6x + 8y = 0$.

5. $x^2 + y^2 - 2x = 3$
$x^2 - 2x + 1 + y^2 = 4$
$(x - 1)^2 + y^2 = 4$
centre: $(1, 0)$; radius 2.

6. $x^2 + y^2 + 4y = 0$
$x^2 + y^2 + 4y + 4 = 4$
$x^2 + (y + 2)^2 = 4$
centre: $(0, -2)$; radius 2.

7. $x^2 + y^2 - 2x + 4y = 4$
$x^2 - 2x + 1 + y^2 + 4y + 4 = 9$
$(x - 1)^2 + (y + 2)^2 = 9$
centre: $(1, -2)$; radius 3.

8. $x^2 + y^2 - 2x - y + 1 = 0$
$x^2 - 2x + 1 + y^2 - y + \frac{1}{4} = \frac{1}{4}$
$(x - 1)^2 + \left(y - \frac{1}{2}\right)^2 = \frac{1}{4}$
centre: $(1, 1/2)$; radius $1/2$.

9. $x^2 + y^2 > 1$ represents all points lying outside the circle of radius 1 centred at the origin.

10. $x^2 + y^2 < 4$ represents the open disk consisting of all points lying inside the circle of radius 2 centred at the origin.

11. $(x + 1)^2 + y^2 \le 4$ represents the closed disk consisting of all points lying inside or on the circle of radius 2 centred at the point $(-1, 0)$.

12. $x^2 + (y - 2)^2 \le 4$ represents the closed disk consisting of all points lying inside or on the circle of radius 2 centred at the point $(0, 2)$.

13. Together, $x^2 + y^2 > 1$ and $x^2 + y^2 < 4$ represent annulus (washer-shaped region) consisting of all points that are outside the circle of radius 1 centred at the origin and inside the circle of radius 2 centred at the origin.

14. Together, $x^2 + y^2 \le 4$ and $(x + 2)^2 + y^2 \le 4$ represent the region consisting of all points that are inside or on both the circle of radius 2 centred at the origin and the circle of radius 2 centred at $(-2, 0)$.

15. Together, $x^2 + y^2 < 2x$ and $x^2 + y^2 < 2y$ (or, equivalently, $(x - 1)^2 + y^2 < 1$ and $x^2 + (y - 1)^2 < 1$) represent the region consisting of all points that are inside both the circle of radius 1 centred at $(1, 0)$ and the circle of radius 1 centred at $(0, 1)$.

16. $x^2 + y^2 - 4x + 2y > 4$ can be rewritten $(x - 2)^2 + (y + 1)^2 > 9$. This equation, taken together with $x + y > 1$, represents all points that lie both outside the circle of radius 3 centred at $(2, -1)$ and above the line $x + y = 1$.

17. The interior of the circle with centre $(-1, 2)$ and radius $\sqrt{6}$ is given by $(x + 1)^2 + (y - 2)^2 < 6$, or $x^2 + y^2 + 2x - 4y < 1$.

18. The exterior of the circle with centre $(2, -3)$ and radius 4 is given by $(x - 2)^2 + (y + 3)^2 > 16$, or $x^2 + y^2 - 4x + 6y > 3$.

19. $x^2 + y^2 < 2, \quad x \ge 1$

20. $x^2 + y^2 > 4, \quad (x - 1)^2 + (y - 3)^2 < 10$

21. The parabola with focus $(0, 4)$ and directrix $y = -4$ has equation $x^2 = 16y$.

22. The parabola with focus $(0, -1/2)$ and directrix $y = 1/2$ has equation $x^2 = -2y$.

23. The parabola with focus $(2, 0)$ and directrix $x = -2$ has equation $y^2 = 8x$.

24. The parabola with focus $(-1, 0)$ and directrix $x = 1$ has equation $y^2 = -4x$.

25. $y = x^2/2$ has focus $(0, 1/2)$ and directrix $y = -1/2$.

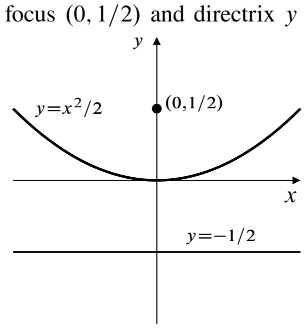


Fig. P.3-25

26. $y = -x^2$ has focus $(0, -1/4)$ and directrix $y = 1/4$.

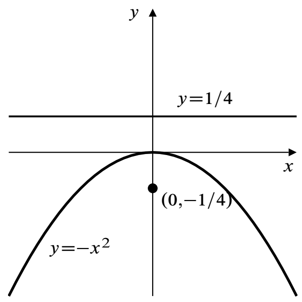


Fig. P.3-26

27. $x = -y^2/4$ has focus $(-1, 0)$ and directrix $x = 1$.

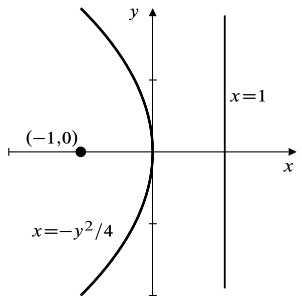


Fig. P.3-27

28. $x = y^2/16$ has focus $(4, 0)$ and directrix $x = -4$.

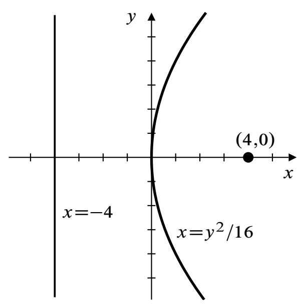


Fig. P.3-28

29.

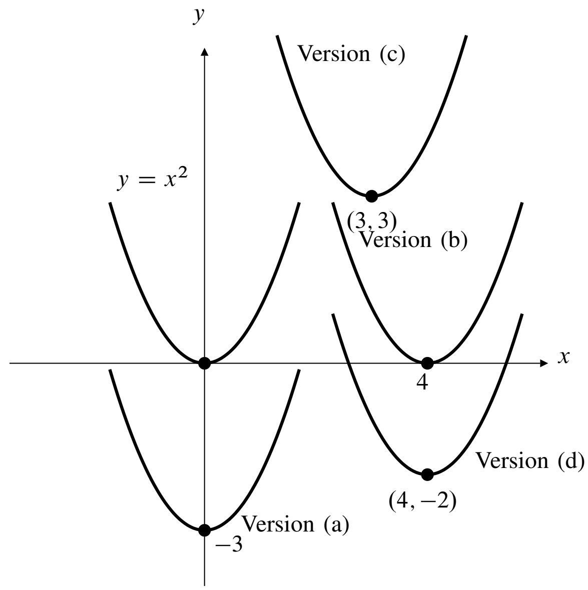


Fig. P.3-29

a) has equation $y = x^2 - 3$.

b) has equation $y = (x-4)^2$ or $y = x^2 - 8x + 16$.

c) has equation $y = (x-3)^2 + 3$ or $y = x^2 - 6x + 12$.

d) has equation $y = (x-4)^2 - 2$, or $y = x^2 - 8x + 14$.

30. a) If $y = mx$ is shifted to the right by amount x_1, the equation $y = m(x - x_1)$ results. If (a, b) satisfies this equation, then $b = m(a - x_1)$, and so $x_1 = a - (b/m)$. Thus the shifted equation is
$y = m(x - a + (b/m)) = m(x - a) + b$.

b) If $y = mx$ is shifted vertically by amount y_1, the equation $y = mx + y_1$ results. If (a, b) satisfies this equation, then $b = ma + y_1$, and so $y_1 = b - ma$. Thus the shifted equation is $y = mx + b - ma = m(x - a) + b$, the same equation obtained in part (a).

31. $y = \sqrt{(x/3) + 1}$

32. $4y = \sqrt{x + 1}$

33. $y = \sqrt{(3x/2) + 1}$

34. $(y/2) = \sqrt{4x + 1}$

35. $y = 1 - x^2$ shifted down 1, left 1 gives $y = -(x + 1)^2$.

36. $x^2 + y^2 = 5$ shifted up 2, left 4 gives
$(x + 4)^2 + (y - 2)^2 = 5$.

37. $y = (x - 1)^2 - 1$ shifted down 1, right 1 gives
$y = (x - 2)^2 - 2$.

38. $y = \sqrt{x}$ shifted down 2, left 4 gives $y = \sqrt{x + 4} - 2$.

39. $y = x^2 + 3$, $y = 3x + 1$. Subtracting these equations gives
$x^2 - 3x + 2 = 0$, or $(x-1)(x-2) = 0$. Thus $x = 1$ or $x = 2$. The corresponding values of y are 4 and 7. The intersection points are $(1, 4)$ and $(2, 7)$.

40. $y = x^2 - 6$, $y = 4x - x^2$. Subtracting these equations gives
$2x^2 - 4x - 6 = 0$, or $2(x-3)(x+1) = 0$. Thus $x = 3$ or $x = -1$. The corresponding values of y are 3 and -5. The intersection points are $(3, 3)$ and $(-1, -5)$.

41. $x^2 + y^2 = 25$, $3x + 4y = 0$. The second equation says that $y = -3x/4$. Substituting this into the first equation gives $25x^2/16 = 25$, so $x = \pm 4$. If $x = 4$, then the second equation gives $y = -3$; if $x = -4$, then $y = 3$. The intersection points are $(4, -3)$ and $(-4, 3)$. Note that having found values for x, we substituted them into the linear equation rather than the quadratic equation to find the corresponding values of y. Had we substituted into the quadratic equation we would have got more solutions (four points in all), but two of them would have failed to satisfy $3x + 4y = 12$. When solving systems of nonlinear equations you should always verify that the solutions you find do satisfy the given equations.

42. $2x^2 + 2y^2 = 5$, $xy = 1$. The second equation says that $y = 1/x$. Substituting this into the first equation gives $2x^2 + (2/x^2) = 5$, or $2x^4 - 5x^2 + 2 = 0$. This equation factors to $(2x^2 - 1)(x^2 - 2) = 0$, so its solutions are $x = \pm 1/\sqrt{2}$ and $x = \pm\sqrt{2}$. The corresponding values of y are given by $y = 1/x$. Therefore, the intersection points are $(1/\sqrt{2}, \sqrt{2})$, $(-1/\sqrt{2}, -\sqrt{2})$, $(\sqrt{2}, 1/\sqrt{2})$, and $(-\sqrt{2}, -1/\sqrt{2})$.

43. $(x^2/4) + y^2 = 1$ is an ellipse with major axis between $(-2, 0)$ and $(2, 0)$ and minor axis between $(0, -1)$ and $(0, 1)$.

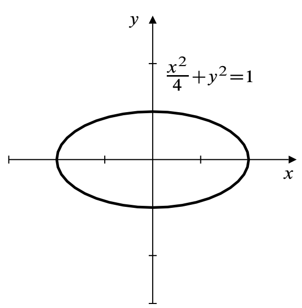


Fig. P.3-43

44. $9x^2 + 16y^2 = 144$ is an ellipse with major axis between $(-4, 0)$ and $(4, 0)$ and minor axis between $(0, -3)$ and $(0, 3)$.

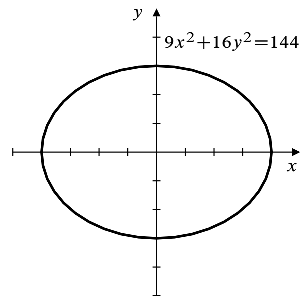


Fig. P.3-44

45. $\dfrac{(x-3)^2}{9} + \dfrac{(y+2)^2}{4} = 1$ is an ellipse with centre at $(3, -2)$, major axis between $(0, -2)$ and $(6, -2)$ and minor axis between $(3, -4)$ and $(3, 0)$.

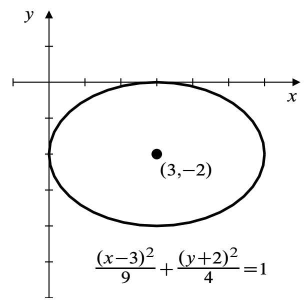


Fig. P.3-45

46. $(x-1)^2 + \dfrac{(y+1)^2}{4} = 4$ is an ellipse with centre at $(1, -1)$, major axis between $(1, -5)$ and $(1, 3)$ and minor axis between $(-1, -1)$ and $(3, -1)$.

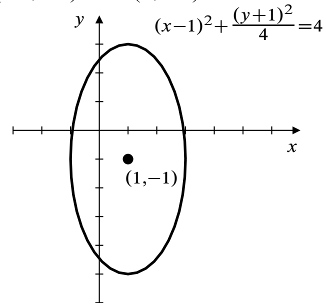


Fig. P.3-46

47. $(x^2/4) - y^2 = 1$ is a hyperbola with centre at the origin and passing through $(\pm 2, 0)$. Its asymptotes are $y = \pm x/2$.

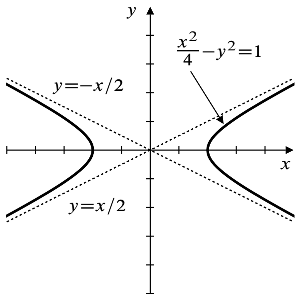


Fig. P.3-47

48. $x^2 - y^2 = -1$ is a rectangular hyperbola with centre at the origin and passing through $(0, \pm 1)$. Its asymptotes are $y = \pm x$.

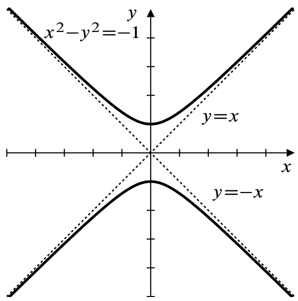


Fig. P.3-48

49. $xy = -4$ is a rectangular hyperbola with centre at the origin and passing through $(2, -2)$ and $(-2, 2)$. Its asymptotes are the coordinate axes.

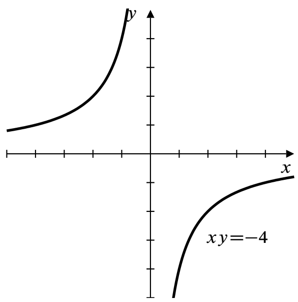


Fig. P.3-49

50. $(x-1)(y+2) = 1$ is a rectangular hyperbola with centre at $(1, -2)$ and passing through $(2, -1)$ and $(0, -3)$. Its asymptotes are $x = 1$ and $y = -2$.

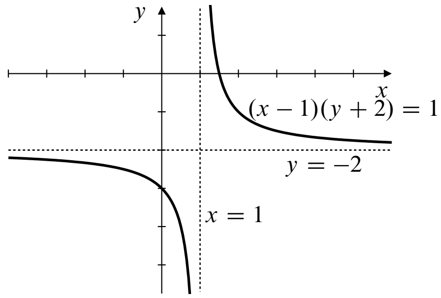


Fig. P.3-50

51. a) Replacing x with $-x$ replaces a graph with its reflection across the y-axis.

b) Replacing y with $-y$ replaces a graph with its reflection across the x-axis.

52. Replacing x with $-x$ and y with $-y$ reflects the graph in both axes. This is equivalent to rotating the graph 180° about the origin.

53. $|x| + |y| = 1$.
In the first quadrant the equation is $x + y = 1$.
In the second quadrant the equation is $-x + y = 1$.
In the third quadrant the equation is $-x - y = 1$.
In the fourth quadrant the equation is $x - y = 1$.

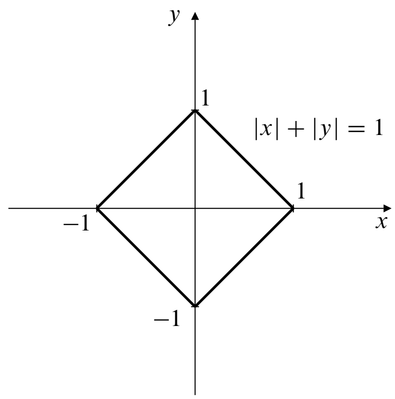


Fig. P.3-53

Section P.4 Functions and Their Graphs (page 32)

1. $f(x) = 1 + x^2$; domain $\mathbb{R}$, range $[1, \infty)$

2. $f(x) = 1 - \sqrt{x}$; domain $[0, \infty)$, range $(-\infty, 1]$

3. $G(x) = \sqrt{8 - 2x}$; domain $(-\infty, 4]$, range $[0, \infty)$

4. $F(x) = 1/(x-1)$; domain $(-\infty, 1) \cup (1, \infty)$, range $(-\infty, 0) \cup (0, \infty)$

5. $h(t) = \dfrac{t}{\sqrt{2-t}}$; domain $(-\infty, 2)$, range $\mathbb{R}$. (The equation $y = h(t)$ can be squared and rewritten as $t^2 + y^2 t - 2y^2 = 0$, a quadratic equation in t having real solutions for every real value of y. Thus the range of h contains all real numbers.)

6. $g(x) = \dfrac{1}{1-\sqrt{x-2}}$; domain $[2, 3) \cup (3, \infty)$, range $(-\infty, 0) \cup (0, \infty)$. The equation $y = g(x)$ can be solved for $x = 2-(1-(1/y))^2$ so has a real solution provided $y \neq 0$.

7.

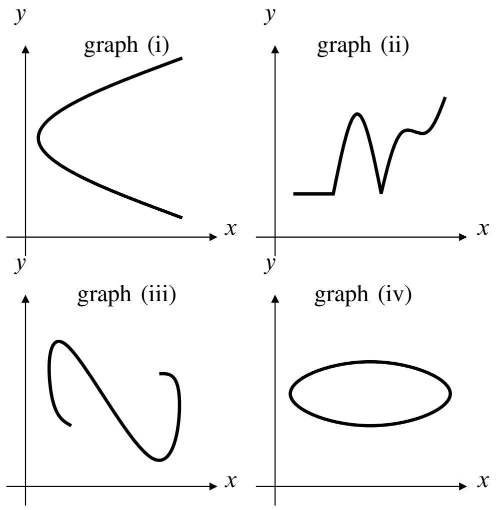


Fig. P.4-7

Graph (ii) is the graph of a function because vertical lines can meet the graph only once. Graphs (i), (iii), and (iv) do not have this property, so are not graphs of functions.

8.

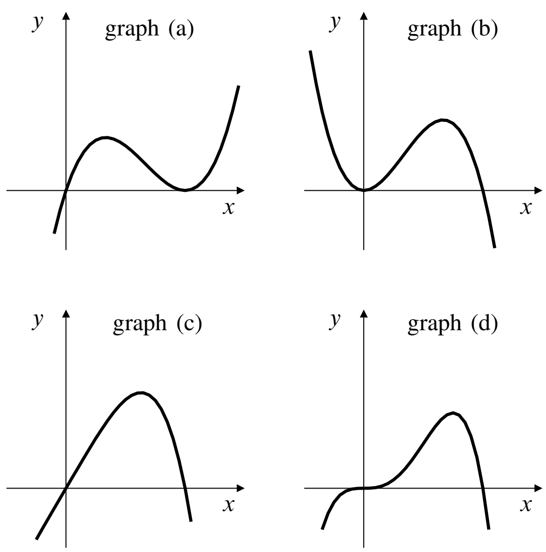


Fig. P.4-8

a) is the graph of $x(1-x)^2$, which is positive for $x > 0$.

b) is the graph of $x^2 - x^3 = x^2(1-x)$, which is positive if $x < 1$.

c) is the graph of $x - x^4$, which is positive if $0 < x < 1$ and behaves like x near 0.

d) is the graph of $x^3 - x^4$, which is positive if $0 < x < 1$ and behaves like x^3 near 0.

9.

x	$f(x) = x^4$
0	0
± 0.5	0.0625
± 1	1
± 1.5	5.0625
± 2	16

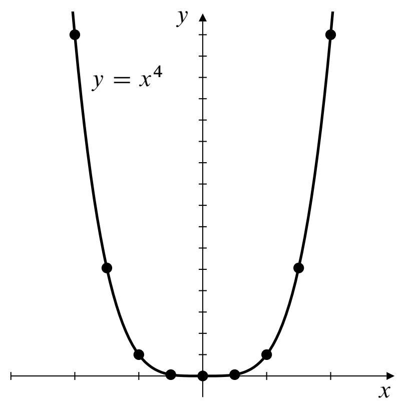


Fig. P.4-9

10.

x	$f(x) = x^{2/3}$
0	0
± 0.5	0.62996
± 1	1
± 1.5	1.3104
± 2	1.5874

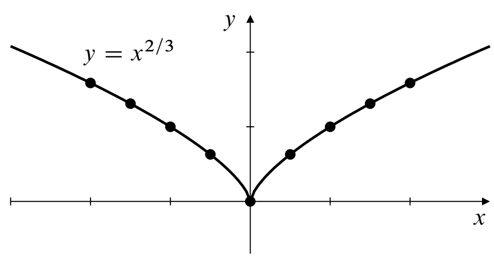


Fig. P.4-10

11. $f(x) = x^2 + 1$ is even: $f(-x) = f(x)$

12. $f(x) = x^3 + x$ is odd: $f(-x) = -f(x)$

13. $f(x) = \dfrac{x}{x^2-1}$ is odd: $f(-x) = -f(x)$

14. $f(x) = \dfrac{1}{x^2 - 1}$ is even: $f(-x) = f(x)$

15. $f(x) = \dfrac{1}{x - 2}$ is odd about $(2, 0)$: $f(2 - x) = -f(2 + x)$

16. $f(x) = \dfrac{1}{x + 4}$ is odd about $(-4, 0)$:
$f(-4 - x) = -f(-4 + x)$

17. $f(x) = x^2 - 6x$ is even about $x = 3$: $f(3 - x) = f(3 + x)$

18. $f(x) = x^3 - 2$ is odd about $(0, -2)$:
$f(-x) + 2 = -(f(x) + 2)$

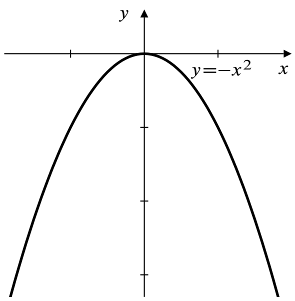
19. $f(x) = |x^3| = |x|^3$ is even: $f(-x) = f(x)$

20. $f(x) = |x + 1|$ is even about $x = -1$:
$f(-1 - x) = f(-1 + x)$

21. $f(x) = \sqrt{2x}$ has no symmetry.

22. $f(x) = \sqrt{(x - 1)^2}$ is even about $x = 1$:
$f(1 - x) = f(1 + x)$

23.



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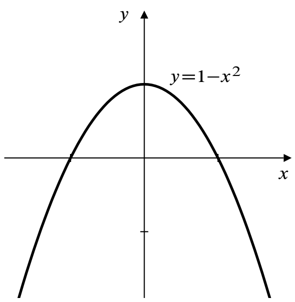

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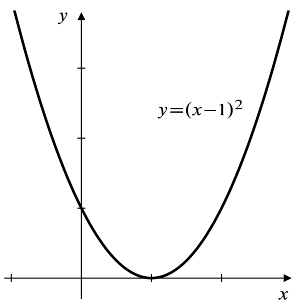

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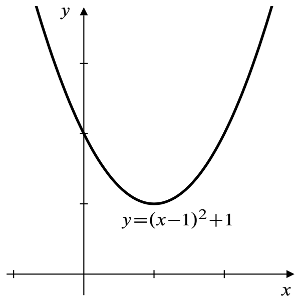

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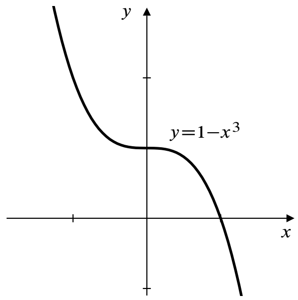

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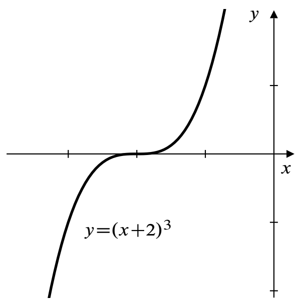

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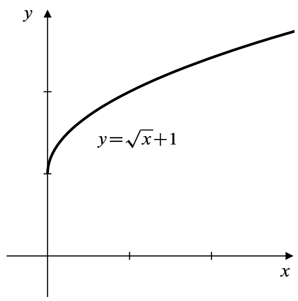

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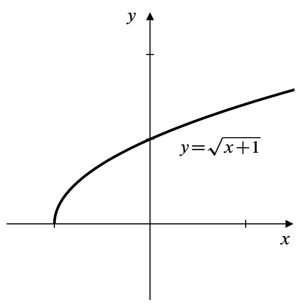

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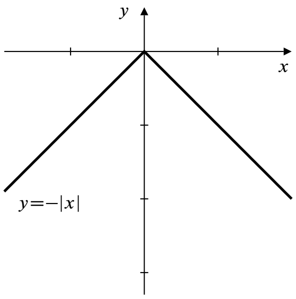

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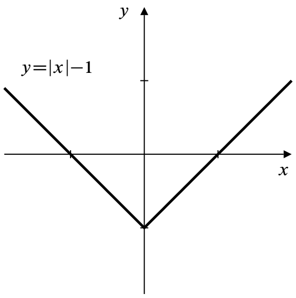

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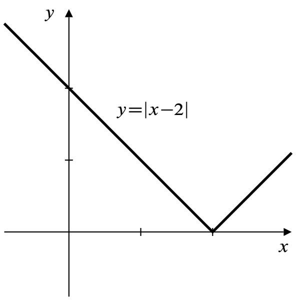

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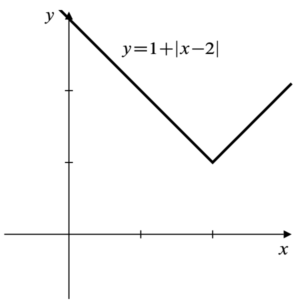

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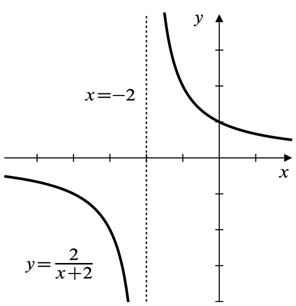

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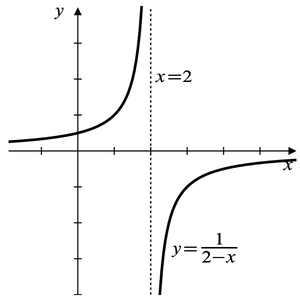

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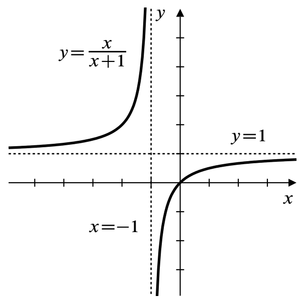

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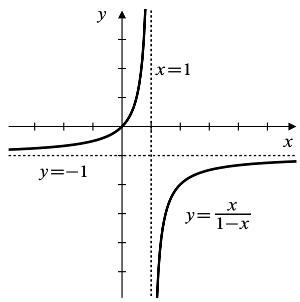

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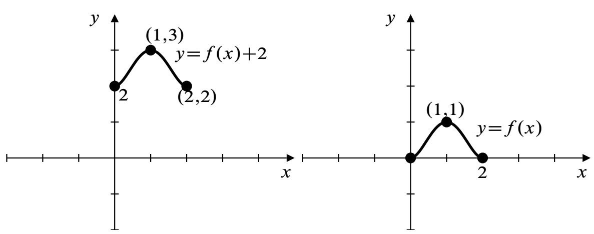


Fig. P.4.39(a) Fig. P.4.39(b)

40.

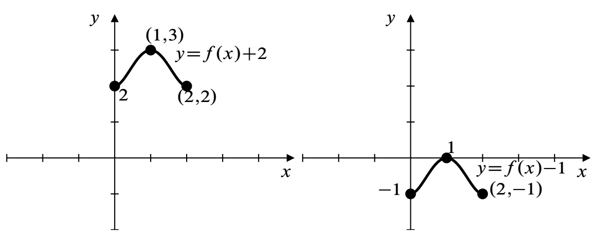
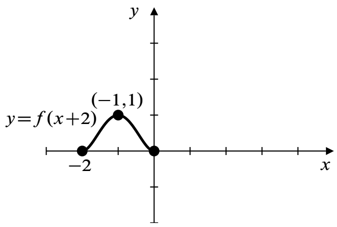


Fig. P.4.40(a) Fig. P.4.40(b)

41.



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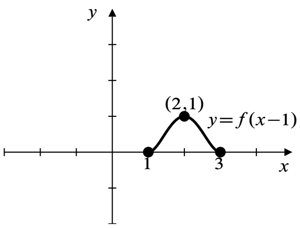

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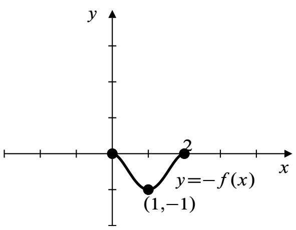

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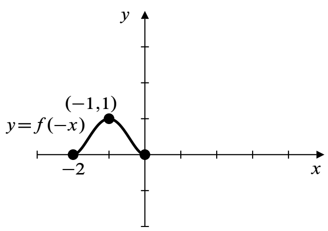

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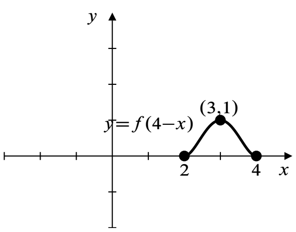

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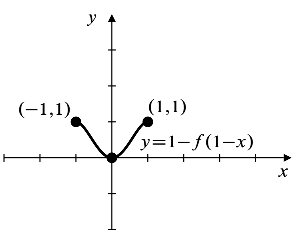


47. Range is approximately $[-0.18, 0.68]$.

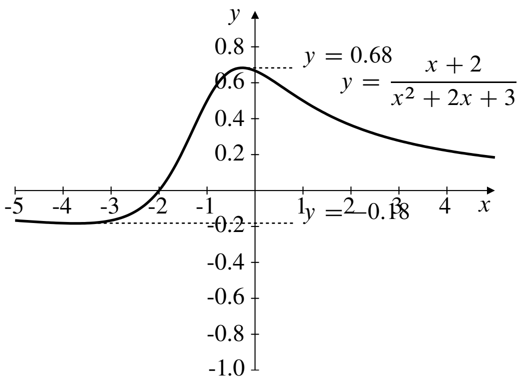


Fig. P.4-47

48. Range is approximately $(-\infty, 0.1] \cup [2.9, \infty)$.

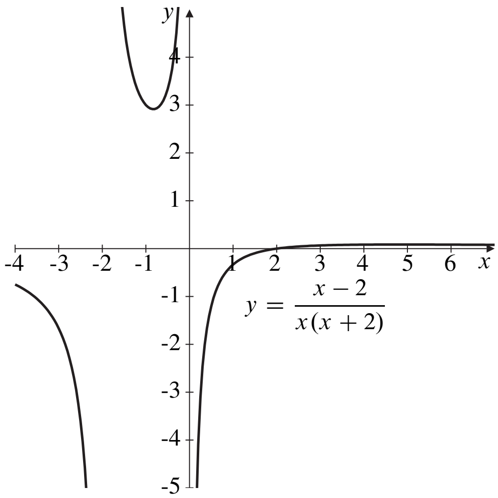


Fig. P.4-48

49.

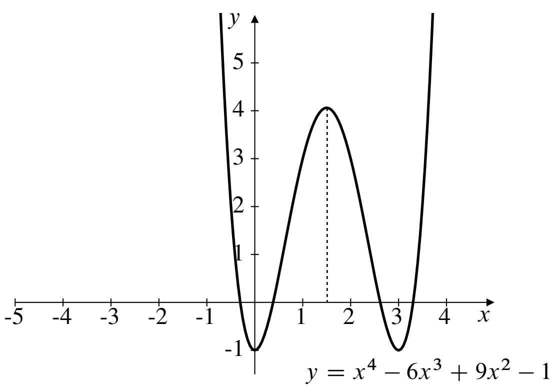


Fig. P.4-49

Apparent symmetry about $x = 1.5$.
This can be confirmed by calculating $f(3-x)$, which turns out to be equal to $f(x)$.

50.

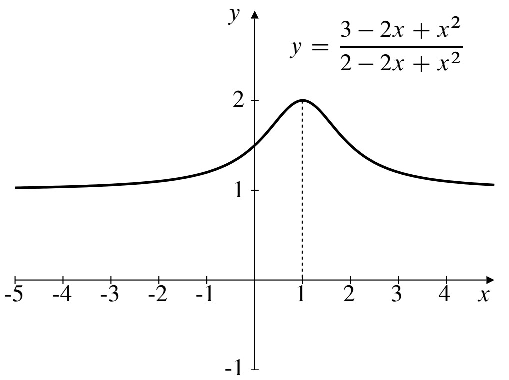


Fig. P.4-50

Apparent symmetry about $x = 1$.
This can be confirmed by calculating $f(2-x)$, which turns out to be equal to $f(x)$.

51.

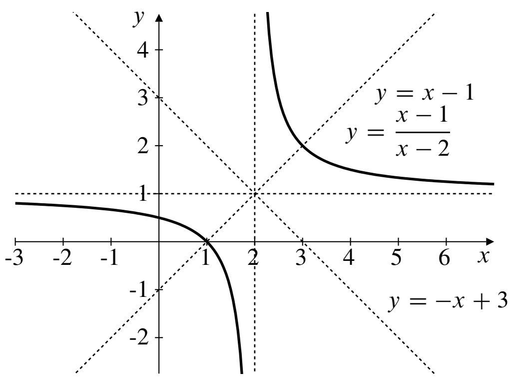


Fig. P.4-51

Apparent symmetry about $(2, 1)$, and about the lines $y = x - 1$ and $y = 3 - x$.

These can be confirmed by noting that $f(x) = 1 + \dfrac{1}{x-2}$, so the graph is that of $1/x$ shifted right 2 units and up one.

52.

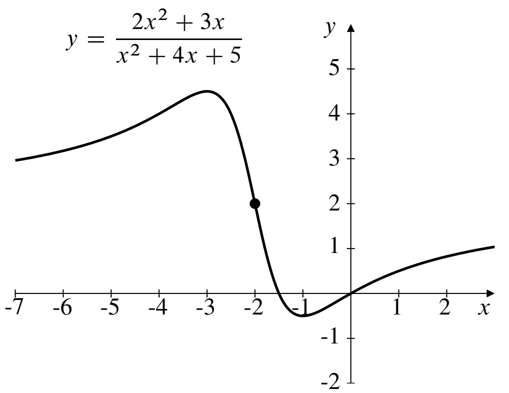


Fig. P.4-52

Apparent symmetry about $(-2, 2)$.
This can be confirmed by calculating shifting the graph right by 2 (replace x with $x-2$) and then down 2 (subtract 2). The result is $-5x/(1 + x^2)$, which is odd.

53. If f is both even and odd the $f(x) = f(-x) = -f(x)$, so $f(x) = 0$ identically.

Section P.5 Combining Functions to Make New Functions (page 38)

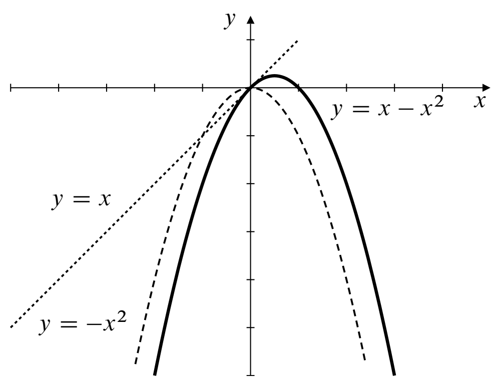
1. $f(x) = x$, $g(x) = \sqrt{x-1}$.
$\mathcal{D}(f) = \mathbb{R}$, $\mathcal{D}(g) = [1, \infty)$.
$\mathcal{D}(f+g) = \mathcal{D}(f-g) = \mathcal{D}(fg) = \mathcal{D}(g/f) = [1, \infty)$,
$\mathcal{D}(f/g) = (1, \infty)$.

$$(f+g)(x) = x + \sqrt{x-1}$$
$$(f-g)(x) = x - \sqrt{x-1}$$
$$(fg)(x) = x\sqrt{x-1}$$
$$(f/g)(x) = x/\sqrt{x-1}$$
$$(g/f)(x) = (\sqrt{1-x})/x$$

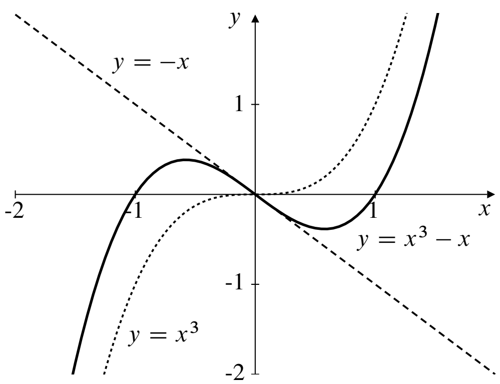
2. $f(x) = \sqrt{1-x}$, $g(x) = \sqrt{1+x}$.
$\mathcal{D}(f) = (-\infty, 1]$, $\mathcal{D}(g) = [-1, \infty)$.
$\mathcal{D}(f+g) = \mathcal{D}(f-g) = \mathcal{D}(fg) = [-1, 1]$,
$\mathcal{D}(f/g) = (-1, 1]$, $\mathcal{D}(g/f) = [-1, 1)$.

$$(f+g)(x) = \sqrt{1-x} + \sqrt{1+x}$$
$$(f-g)(x) = \sqrt{1-x} - \sqrt{1+x}$$
$$(fg)(x) = \sqrt{1-x^2}$$
$$(f/g)(x) = \sqrt{(1-x)/(1+x)}$$
$$(g/f)(x) = \sqrt{(1+x)/(1-x)}$$

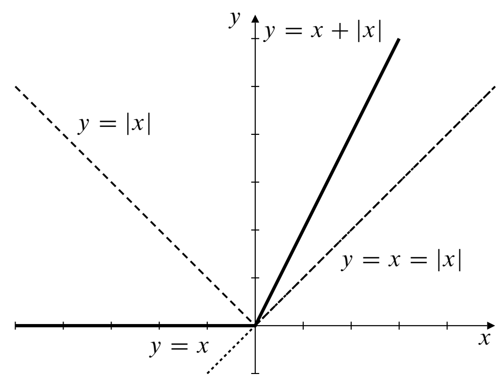
3.



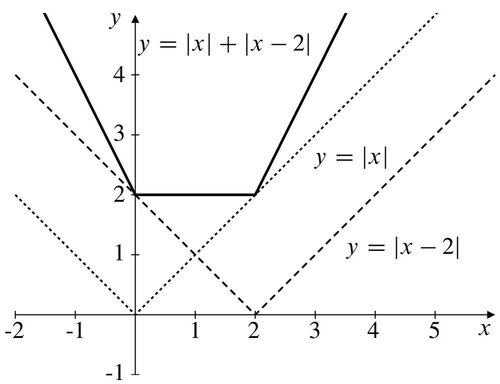
4.



5.



6.



7. $f(x) = x + 5,\ g(x) = x^2 - 3.$
$f \circ g(0) = f(-3) = 2, \quad g(f(0)) = g(5) = 22$
$f(g(x)) = f(x^2 - 3) = x^2 + 2$
$g \circ f(x) = g(f(x)) = g(x + 5) = (x + 5)^2 - 3$
$f \circ f(-5) = f(0) = 5, \quad g(g(2)) = g(1) = -2$
$f(f(x)) = f(x + 5) = x + 10$
$g \circ g(x) = g(g(x)) = (x^2 - 3)^2 - 3$

8. $f(x) = 2/x,\ g(x) = x/(1 - x).$
$f \circ f(x) = 2/(2/x) = x; \quad \mathcal{D}(f \circ f) = \{x : x \neq 0\}$
$f \circ g(x) = 2/(x/(1 - x)) = 2(1 - x)/x;$
$\mathcal{D}(f \circ g) = \{x : x \neq 0, 1\}$
$g \circ f(x) = (2/x)/(1 - (2/x)) = 2/(x - 2);$
$\mathcal{D}(g \circ f) = \{x : x \neq 0, 2\}$
$g \circ g(x) = (x/(1 - x))/(1 - (x/(1 - x))) = x/(1 - 2x);$
$\mathcal{D}(g \circ g) = \{x : x \neq 1/2, 1\}$

9. $f(x) = 1/(1 - x),\ g(x) = \sqrt{x - 1}.$
$f \circ f(x) = 1/(1 - (1/(1 - x))) = (x - 1)/x;$
$\mathcal{D}(f \circ f) = \{x : x \neq 0, 1\}$
$f \circ g(x) = 1/(1 - \sqrt{x - 1});$
$\mathcal{D}(f \circ g) = \{x : x \geq 1,\ x \neq 2\}$
$g \circ f(x) = \sqrt{(1/(1 - x)) - 1} = \sqrt{x/(1 - x)};$
$\mathcal{D}(g \circ f) = [0, 1)$
$g \circ g(x) = \sqrt{\sqrt{x - 1} - 1}; \quad \mathcal{D}(g \circ g) = [2, \infty)$

10. $f(x) = (x + 1)/(x - 1) = 1 + 2/(x - 1),\ g(x) = \text{sgn}\,(x).$
$f \circ f(x) = 1 + 2/(1 + (2/(x - 1) - 1)) = x;$
$\mathcal{D}(f \circ f) = \{x : x \neq 1\}$
$f \circ g(x) = \dfrac{\text{sgn}\,x + 1}{\text{sgn}\,x - 1} = 0; \quad \mathcal{D}(f \circ g) = (-\infty, 0)$
$g \circ f(x) = \text{sgn}\left(\dfrac{x + 1}{x - 1}\right) = \begin{cases} 1 & \text{if } x < -1 \text{ or } x > 1 \\ -1 & \text{if } -1 < x < 1 \end{cases};$
$\mathcal{D}(g \circ f) = \{x : x \neq -1,\ 1\}$
$g \circ g(x) = \text{sgn}\,(\text{sgn}\,(x)) = \text{sgn}\,(x); \quad \mathcal{D}(g \circ g) = \{x : x \neq 0\}$

	$f(x)$	$g(x)$	$f \circ g(x)$
11.	x^2	$x + 1$	$(x + 1)^2$
12.	$x - 4$	$x + 4$	x
13.	$\sqrt{x}$	x^2	$\lvert x\rvert$
14.	$2x^3 + 3$	$x^{1/3}$	$2x + 3$
15.	$(x + 1)/x$	$1/(x - 1)$	x
16.	$1/(x + 1)^2$	$x - 1$	$1/x^2$

17. $y = \sqrt{x}.$
$y = 2 + \sqrt{x}$: previous graph is raised 2 units.
$y = 2 + \sqrt{3 + x}$: previous graph is shiftend left 3 units.
$y = 1/(2 + \sqrt{3 + x})$: previous graph turned upside down and shrunk vertically.

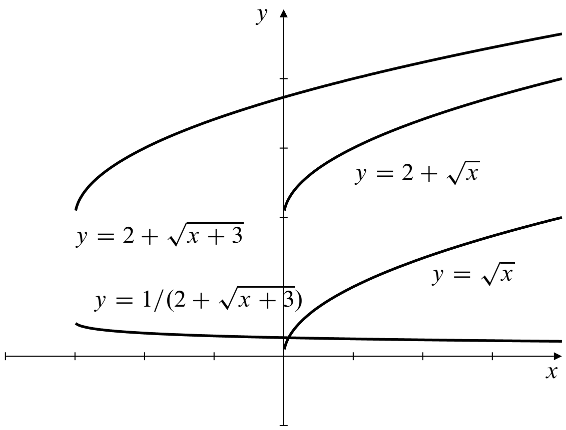


Fig. P.5-17

18.

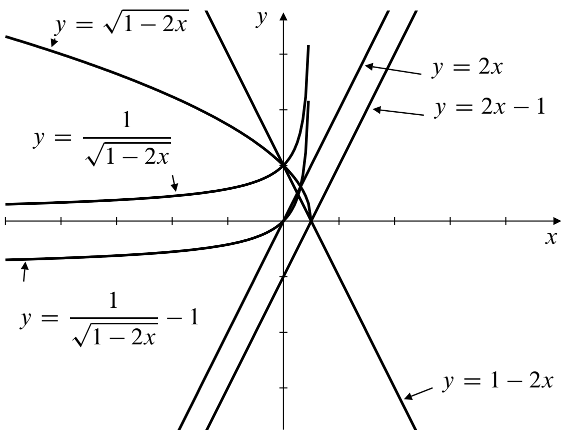
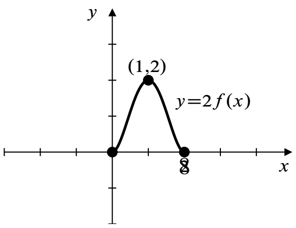


Fig. P.5-18

19.



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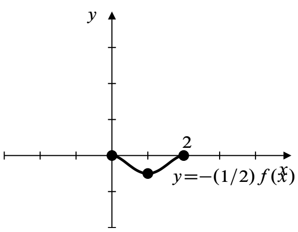

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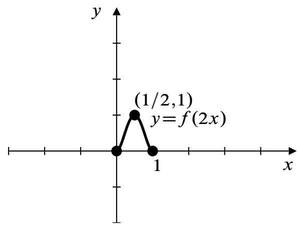

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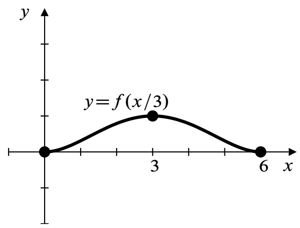

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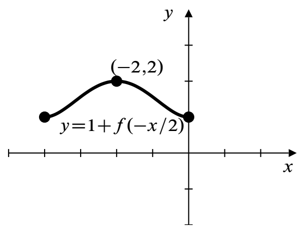

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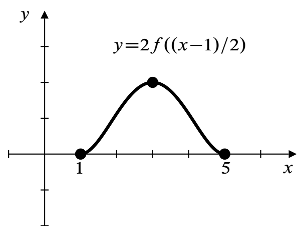

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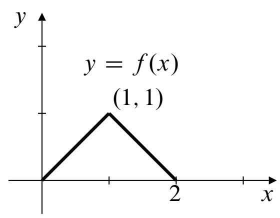

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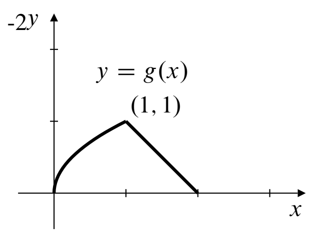


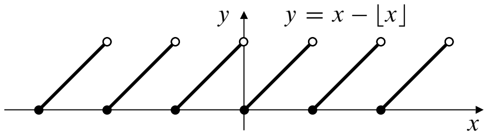
27. $F(x) = Ax + B$
(a) $F \circ F(x) = F(x)$
$\Rightarrow \quad A(Ax + B) + B = Ax + B$
$\Rightarrow \quad A[(A-1)x + B] = 0$
Thus, either $A = 0$ or $A = 1$ and $B = 0$.
(b) $F \circ F(x) = x$
$\Rightarrow \quad A(Ax + B) + B = x$
$\Rightarrow \quad (A^2 - 1)x + (A+1)B = 0$
Thus, either $A = -1$ or $A = 1$ and $B = 0$

28. $\lfloor x \rfloor = 0$ for $0 \le x < 1$; $\lceil x \rceil = 0$ for $-1 \le x < 0$.

29. $\lfloor x \rfloor = \lceil x \rceil$ for all integers x.

30. $\lceil -x \rceil = -\lfloor x \rfloor$ is true for all real x; if $x = n + y$ where n is an integer and $0 \le y < 1$, then $-x = -n - y$, so that $\lceil -x \rceil = -n$ and $\lfloor x \rfloor = n$.

31.



32. $f(x)$ is called the integer part of x because $|f(x)|$ is the largest integer that does not exceed x; i.e. $|x| = |f(x)| + y$, where $0 \le y < 1$.

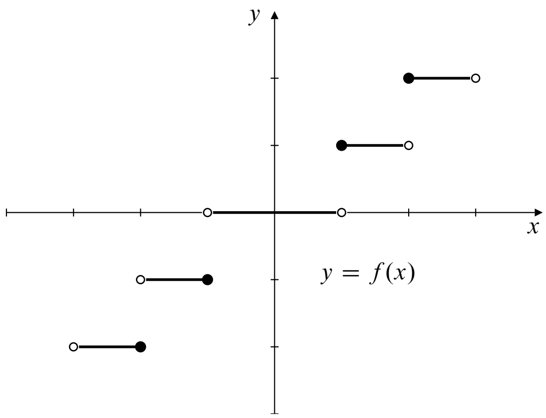


Fig. P.5-32

33. If f is even and g is odd, then: f^2, g^2, $f \circ g$, $g \circ f$, and $f \circ f$ are all even. fg, f/g, g/f, and $g \circ g$ are odd, and $f + g$ is neither even nor odd. Here are two typical verifications:

$$f \circ g(-x) = f(g(-x)) = f(-g(x)) = f(g(x)) = f \circ g(x)$$
$$(fg)(-x) = f(-x)g(-x) = f(x)[-g(x)]$$
$$= -f(x)g(x) = -(fg)(x).$$

The others are similar.

34. f even $\Leftrightarrow f(-x) = f(x)$
f odd $\Leftrightarrow f(-x) = -f(x)$
f even and odd $\Rightarrow f(x) = -f(x) \Rightarrow 2f(x) = 0$
$\Rightarrow f(x) = 0$

35. a) Let $E(x) = \frac{1}{2}[f(x) + f(-x)]$.
Then $E(-x) = \frac{1}{2}[f(-x) + f(x)] = E(x)$. Hence, $E(x)$ is even.
Let $O(x) = \frac{1}{2}[f(x) - f(-x)]$.
Then $O(-x) = \frac{1}{2}[f(-x) - f(x)] = -O(x)$ and $O(x)$ is odd.

$$\begin{aligned} E(x) &+ O(x) \\ &= \tfrac{1}{2}[f(x) + f(-x)] + \tfrac{1}{2}[f(x) - f(-x)] \\ &= f(x). \end{aligned}$$

Hence, $f(x)$ is the sum of an even function and an odd function.

b) If $f(x) = E_1(x) + O_1(x)$ where E_1 is even and O_1 is odd, then

$$E_1(x) + O_1(x) = f(x) = E(x) + O(x).$$

Thus $E_1(x) - E(x) = O(x) - O_1(x)$. The left side of this equation is an even function and the right side is an odd function. Hence both sides are both even and odd, and are therefore identically 0 by Exercise 36. Hence $E_1 = E$ and $O_1 = O$. This shows that f can be written in only one way as the sum of an even function and an odd function.

Section P.6 Polynomials and Rational Functions (page 45)

1. $x^2 - 7x + 10 = (x+5)(x+2)$
The roots are -5 and -2.

2. $x^2 - 3x - 10 = (x-5)(x+2)$
The roots are 5 and -2.

3. If $x^2 + 2x + 2 = 0$, then $x = \dfrac{-2 \pm \sqrt{4-8}}{2} = -1 \pm i$.
The roots are $-1 + i$ and $-1 - i$.
$x^2 + 2x + 2 = (x + 1 - i)(x + 1 + i)$.

4. Rather than use the quadratic formula this time, let us complete the square.

$$\begin{aligned} x^2 - 6x + 13 &= x^2 - 6x + 9 + 4 \\ &= (x-3)^2 + 2^2 \\ &= (x - 3 - 2i)(x - 3 + 2i). \end{aligned}$$

The roots are $3 + 2i$ and $3 - 2i$.

5. $16x^4 - 8x^2 + 1 = (4x^2 - 1)^2 = (2x-1)^2(2x+1)^2$. There are two double roots: $1/2$ and $-1/2$.

6. $x^4 + 6x^3 + 9x^2 = x^2(x^2 + 6x + 9) = x^2(x+3)^2$. There are two double roots, 0 and -3.

7. $x^3+1=(x+1)(x^2-x+1)$. One root is -1. The other two are the solutions of $x^2-x+1=0$, namely

$$x=\frac{1\pm\sqrt{1-4}}{2}=\frac{1}{2}\pm\frac{\sqrt{3}}{2}i.$$

We have

$$x^3+1=(x+1)\left(x-\frac{1}{2}-\frac{\sqrt{3}}{2}i\right)\left(x-\frac{1}{2}+\frac{\sqrt{3}}{2}i\right).$$

8. $x^4-1=(x^2-1)(x^2+1)=(x-1)(x+1)(x-i)(x+i)$. The roots are 1, -1, i, and $-i$.

9. $x^6-3x^4+3x^2-1=(x^2-1)^3=(x-1)^3(x+1)^3$. The roots are 1 and -1, each with multiplicity 3.

10.
$$\begin{aligned}x^5-x^4-16x+16&=(x-1)(x^4-16)\\&=(x-1)(x^2-4)(x^4+4)\\&=(x-1)(x-2)(x+2)(x-2i)(x+2i).\end{aligned}$$

The roots are 1, 2, -2, $2i$, and $-2i$.

11.
$$\begin{aligned}x^5+x^3+8x^2+8&=(x^2+1)(x^3+8)\\&=(x+2)(x-i)(x+i)(x^2-2x+4)\end{aligned}$$

Three of the five roots are -2, i and $-i$. The remaining two are solutions of $x^2-2x+4=0$, namely $x=\dfrac{2\pm\sqrt{4-16}}{2}=1\pm\sqrt{3}i$. We have

$$x^5+x^3+8x^2+8=(x+2)(x-i)(x+i)(x-a+\sqrt{3}i)(x-a-\sqrt{3}i).$$

12.
$$\begin{aligned}x^9-4x^7-x^6+4x^4&=x^4(x^5-x^2-4x^3+4)\\&=x^4(x^3-1)(x^2-4)\\&=x^4(x-1)(x-2)(x+2)(x^2+x+1).\end{aligned}$$

Seven of the nine roots are: 0 (with multiplicity 4), 1, 2, and -2. The other two roots are solutions of $x^2+x+1=0$, namely

$$x=\frac{-1\pm\sqrt{1-4}}{2}=-\frac{1}{2}\pm\frac{\sqrt{3}}{2}i.$$

The required factorization of $x^9-4x^7-x^6+4x^4$ is

$$x^4(x-1)(x-2)(x+2)\left(x-\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)\left(x-\frac{1}{2}-\frac{\sqrt{3}}{2}i\right).$$

13. The denominator is $x^2+2x+2=(x+1)^2+1$ which is never 0. Thus the rational function is defined for all real numbers.

14. The denominator is $x^3-x=x(x-1)(x+1)$ which is zero if $x=0$, 1, or -1. Thus the rational function is defined for all real numbers except 0, 1, and -1.

15. The denominator is $x^3+x^2=x^2(x+1)$ which is zero only if $x=0$ or $x=-1$. Thus the rational function is defined for all real numbers except 0 and -1.

16. The denominator is x^2+x-1, which is a quadratic polynomial whose roots can be found with the quadratic formula. They are $x=(-1\pm\sqrt{1+4})/2$. Hence the given rational function is defined for all real numbers except $(-1-\sqrt{5})/2$ and $(-1+\sqrt{5})/2$.

17.
$$\begin{aligned}\frac{x^3-1}{x^2-2}&=\frac{x^3-2x+2x-1}{x^2-2}\\&=\frac{x(x^2-2)+2x-1}{x^2-2}\\&=x+\frac{2x-1}{x^2-2}.\end{aligned}$$

18.
$$\begin{aligned}\frac{x^2}{x^2+5x+3}&=\frac{x^2+5x+3-5x-3}{x^2+5x+3}\\&=1+\frac{-5x-3}{x^2+5x+3}.\end{aligned}$$

19.
$$\begin{aligned}\frac{x^3}{x^2+2x+3}&=\frac{x^3+2x^2+3x-2x^2-3x}{x^2+2x+3}\\&=\frac{x(x^2+2x+3)-2x^2-3x}{x^2+2x+3}\\&=x-\frac{2(x^2+2x+3)-4x-6+3x}{x^2+2x+3}\\&=x-2+\frac{x+6}{x^2+2x+3}.\end{aligned}$$

20.
$$\begin{aligned}\frac{x^4+x^2}{x^3+x^2+1}&=\frac{x(x^3+x^2+1)-x^3-x+x^2}{x^3+x^2+1}\\&=x+\frac{-(x^3+x^2+1)+x^2+1-x+x^2}{x^3+x^2+1}\\&=x-1+\frac{2x^2-x+1}{x^3+x^2+1}.\end{aligned}$$

21. As in Example 6, we want $a^4=4$, so $a^2=2$ and $a=\sqrt{2}$, $b=\pm\sqrt{2}a=\pm 2$. Thus $P(x)=(x^2-2x+2)(x^2+2x+2)$.

22. Following the method of Example 6, we calculate

$$(x^2-bx+a^2)(x^2+bx+a^2)=x^4+a^4+(2a^2-b^2)x^2=x^4+x^2+1$$

provided $a=1$ and $b^2=-1+2a^2=1$, so $b=\pm 1$. Thus $P(x)=(x^2-x+1)(x^2+x+1)$.

23. Let $P(x)=a_nx^n+a_{n-1}x^{n-1}+\cdots+a_1x+a_0$, where $n\ge 1$. By the Factor Theorem, $x-1$ is a factor of $P(x)$ if and only if $P(1)=0$, that is, if and only if $a_n+a_{n-1}+\cdots+a_1+a_0=0$.

24. Let $P(x)=a_nx^n+a_{n-1}x^{n-1}+\cdots+a_1x+a_0$, where $n\ge 1$. By the Factor Theorem, $x+1$ is a factor of $P(x)$ if and only if $P(-1)=0$, that is, if and only if $a_0-a_1+a_2-a_3+\cdots+(-1)^na_n=0$. This condition says that the sum of the coefficients of even powers is equal to the sum of coefficients of odd powers.

25. Let $P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$, where the coefficients a_k, $0 \le k \le n$ are all real numbers, so that $\overline{a_k} = a_k$. Using the facts about conjugates of sums and products mentioned in the statement of the problem, we see that if $z = x + iy$, where x and y are real, then

$$\begin{aligned}\overline{P(z)} &= \overline{a_n z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0}\\ &= a_n \overline{z}^n + a_{n-1}\overline{z}^{n-1} + \cdots + a_1\overline{z} + a_0\\ &= P(\overline{z}).\end{aligned}$$

If z is a root of P, then $P(\overline{z}) = \overline{P(z)} = \overline{0} = 0$, and $\overline{z}$ is also a root of P.

26. By the previous exercise, $\overline{z} = u - iv$ is also a root of P. Therefore $P(x)$ has two linear factors $x - u - iv$ and $x - u + iv$. The product of these factors is the real quadratic factor $(x-u)^2 - i^2v^2 = x^2 - 2ux + u^2 + v^2$, which must also be a factor of $P(x)$.

27. By the previous exercise

$$\frac{P(x)}{x^2 - 2ux + u^2 + v^2} = \frac{P(x)}{(x-u-iv)(x-u+iv)} = Q_1(x),$$

where Q_1, being a quotient of two polynomials with real coefficients, must also have real coefficients. If $z = u + iv$ is a root of P having multiplicity $m > 1$, then it must also be a root of Q_1 (of multiplicity $m-1$), and so, therefore, $\overline{z}$ must be a root of Q_1, as must be the real quadratic $x^2 - 2ux + u^2 + v^2$. Thus

$$\frac{P(x)}{(x^2 - 2ux + u^2 + v^2)^2} = \frac{Q_1(x)}{x^2 - 2ux + u^2 + v^2} = Q_2(x),$$

where Q_2 is a polynomial with real coefficients. We can continue in this way until we get

$$\frac{P(x)}{(x^2 - 2ux + u^2 + v^2)^m} = Q_m(x),$$

where Q_m no longer has z (or $\overline{z}$) as a root. Thus z and $\overline{z}$ must have the same multiplicity as roots of P.

Section P.7 The Trigonometric Functions (page 57)

1. $\cos\left(\frac{3\pi}{4}\right) = \cos\left(\pi - \frac{\pi}{4}\right) = -\cos\frac{\pi}{4} = -\frac{1}{\sqrt{2}}$

2. $\tan\frac{-3\pi}{4} = -\tan\frac{3\pi}{4} = -1$

3. $\sin\frac{2\pi}{3} = \sin\left(\pi - \frac{\pi}{3}\right) = \sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$

4.
$$\begin{aligned}\sin\left(\frac{7\pi}{12}\right) &= \sin\left(\frac{\pi}{4} + \frac{\pi}{3}\right)\\ &= \sin\frac{\pi}{4}\cos\frac{\pi}{3} + \cos\frac{\pi}{4}\sin\frac{\pi}{3}\\ &= \frac{1}{\sqrt{2}}\frac{1}{2} + \frac{1}{\sqrt{2}}\frac{\sqrt{3}}{2} = \frac{1+\sqrt{3}}{2\sqrt{2}}\end{aligned}$$

5.
$$\begin{aligned}\cos\frac{5\pi}{12} &= \cos\left(\frac{2\pi}{3} - \frac{\pi}{4}\right)\\ &= \cos\frac{2\pi}{3}\cos\frac{\pi}{4} + \sin\frac{2\pi}{3}\sin\frac{\pi}{4}\\ &= -\left(\frac{1}{2}\right)\left(\frac{1}{\sqrt{2}}\right) + \left(\frac{\sqrt{3}}{2}\right)\left(\frac{1}{\sqrt{2}}\right)\\ &= \frac{\sqrt{3}-1}{2\sqrt{2}}\end{aligned}$$

6.
$$\begin{aligned}\sin\frac{11\pi}{12} &= \sin\frac{\pi}{12}\\ &= \sin\left(\frac{\pi}{3} - \frac{\pi}{4}\right)\\ &= \sin\frac{\pi}{3}\cos\frac{\pi}{4} - \cos\frac{\pi}{3}\sin\frac{\pi}{4}\\ &= \left(\frac{\sqrt{3}}{2}\right)\left(\frac{1}{\sqrt{2}}\right) - \left(\frac{1}{2}\right)\left(\frac{1}{\sqrt{2}}\right)\\ &= \frac{\sqrt{3}-1}{2\sqrt{2}}\end{aligned}$$

7.
$$\begin{aligned}\cos(\pi + x) &= \cos\Big(2\pi - (\pi - x)\Big)\\ &= \cos\Big(-(\pi - x)\Big)\\ &= \cos(\pi - x) = -\cos x\end{aligned}$$

8. $\sin(2\pi - x) = -\sin x$

9.
$$\begin{aligned}\sin\left(\frac{3\pi}{2} - x\right) &= \sin\left(\pi - \left(x - \frac{\pi}{2}\right)\right)\\ &= \sin\left(x - \frac{\pi}{2}\right)\\ &= -\sin\left(\frac{\pi}{2} - x\right)\\ &= -\cos x\end{aligned}$$

10.
$$\begin{aligned}\cos\left(\frac{3\pi}{2} + x\right) &= \cos\frac{3\pi}{2}\cos x - \sin\frac{3\pi}{2}\sin x\\ &= (-1)(-\sin x) = \sin x\end{aligned}$$

11.
$$\begin{aligned}\tan x + \cot x &= \frac{\sin x}{\cos x} + \frac{\cos x}{\sin x}\\ &= \frac{\sin^2 x + \cos^2 x}{\cos x \sin x}\\ &= \frac{1}{\cos x \sin x}\end{aligned}$$

12. $$\frac{\tan x - \cot x}{\tan x + \cot x} = \frac{\left(\frac{\sin x}{\cos x} - \frac{\cos x}{\sin x}\right)}{\left(\frac{\sin x}{\cos x} + \frac{\cos x}{\sin x}\right)} = \frac{\left(\frac{\sin^2 x - \cos^2 x}{\cos x \sin x}\right)}{\left(\frac{\sin^2 x + \cos^2 x}{\cos x \sin x}\right)} = \sin^2 x - \cos^2 x$$

13. $$\cos^4 x - \sin^4 x = (\cos^2 x - \sin^2 x)(\cos^2 x + \sin^2 x) = \cos^2 x - \sin^2 x = \cos(2x)$$

14. $(1 - \cos x)(1 + \cos x) = 1 - \cos^2 x = \sin^2 x$ implies $\dfrac{1 - \cos x}{\sin x} = \dfrac{\sin x}{1 + \cos x}$. Now

$$\frac{1 - \cos x}{\sin x} = \frac{1 - \cos 2\left(\frac{x}{2}\right)}{\sin 2\left(\frac{x}{2}\right)} = \frac{1 - \left(1 - 2\sin^2\left(\frac{x}{2}\right)\right)}{2\sin\frac{x}{2}\cos\frac{x}{2}} = \frac{\sin\frac{x}{2}}{\cos\frac{x}{2}} = \tan\frac{x}{2}$$

15. $$\frac{1 - \cos x}{1 + \cos x} = \frac{2\sin^2\left(\frac{x}{2}\right)}{2\cos^2\left(\frac{x}{2}\right)} = \tan^2\left(\frac{x}{2}\right)$$

16. $$\frac{\cos x - \sin x}{\cos x + \sin x} = \frac{(\cos x - \sin x)^2}{(\cos x + \sin x)(\cos x - \sin x)} = \frac{\cos^2 x - 2\sin x\cos x + \sin^2 x}{\cos^2 x - \sin^2 x} = \frac{1 - \sin(2x)}{\cos(2x)} = \sec(2x) - \tan(2x)$$

17. $$\begin{aligned}\sin 3x &= \sin(2x + x) \\ &= \sin 2x \cos x + \cos 2x \sin x \\ &= 2\sin x\cos^2 x + \sin x(1 - 2\sin^2 x) \\ &= 2\sin x(1 - \sin^2 x) + \sin x - 2\sin^3 x \\ &= 3\sin x - 4\sin^3 x\end{aligned}$$

18. $$\begin{aligned}\cos 3x &= \cos(2x + x) \\ &= \cos 2x \cos x - \sin 2x \sin x \\ &= (2\cos^2 x - 1)\cos x - 2\sin^2 x\cos x \\ &= 2\cos^3 x - \cos x - 2(1 - \cos^2 x)\cos x \\ &= 4\cos^3 x - 3\cos x\end{aligned}$$

19. $\cos 2x$ has period π.

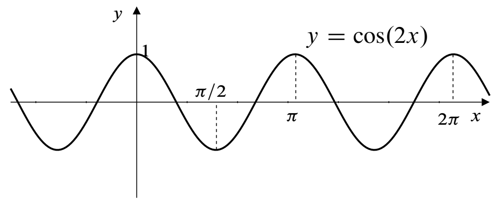


Fig. P.7-19

20. $\sin\dfrac{x}{2}$ has period 4π.

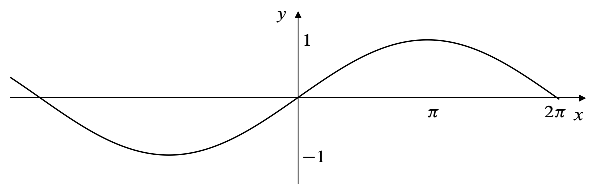


Fig. P.7-20

21. $\sin \pi x$ has period 2.

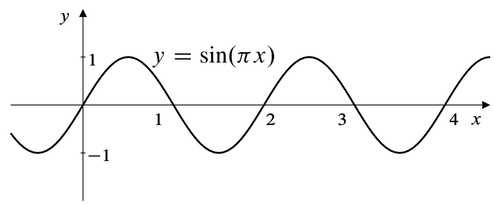


Fig. P.7-21

22. $\cos\dfrac{\pi x}{2}$ has period 4.

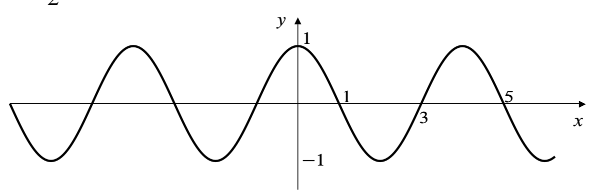
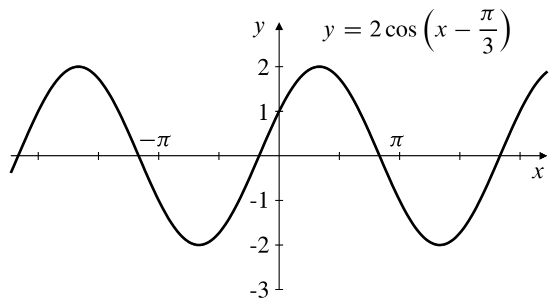
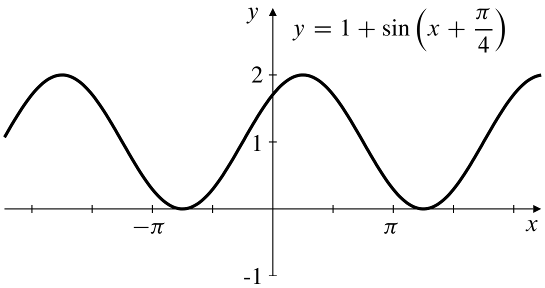


Fig. P.7-22

23.



24.



25. $\sin x = \frac{3}{5}, \quad \frac{\pi}{2} < x < \pi$

$\cos x = -\frac{4}{5}, \quad \tan x = -\frac{3}{4}$

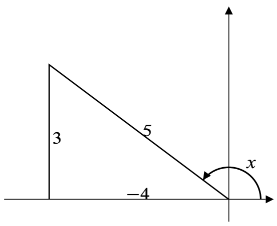


Fig. P.7-25

26. $\tan x = 2$ where x is in $[0, \frac{\pi}{2}]$. Then $\sec^2 x = 1 + \tan^2 x = 1 + 4 = 5$. Hence, $\sec x = \sqrt{5}$ and $\cos x = \frac{1}{\sec x} = \frac{1}{\sqrt{5}}$, $\sin x = \tan x \cos x = \frac{2}{\sqrt{5}}$.

27. $\cos x = \frac{1}{3}, \quad -\frac{\pi}{2} < x < 0$

$\sin x = -\frac{\sqrt{8}}{3} = -\frac{2}{3}\sqrt{2}$

$\tan x = -\frac{\sqrt{8}}{1} = -2\sqrt{2}$

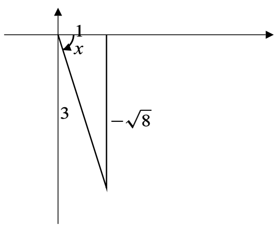


Fig. P.7-27

28. $\cos x = -\frac{5}{13}$ where x is in $\left[\frac{\pi}{2}, \pi\right]$. Hence, $\sin x = \sqrt{1 - \cos^2 x} = \sqrt{1 - \frac{25}{169}} = \frac{12}{13}$, $\tan x = -\frac{12}{5}$.

29. $\sin x = -\frac{1}{2}, \quad \pi < x < \frac{3\pi}{2}$

$\cos x = -\frac{\sqrt{3}}{2}$

$\tan x = \frac{1}{\sqrt{3}}$

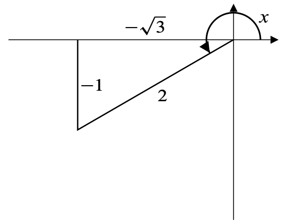


Fig. P.7-29

30. $\tan x = \frac{1}{2}$ where x is in $[\pi, \frac{3\pi}{2}]$. Then, $\sec^2 x = 1 + \frac{1}{4} = \frac{5}{4}$. Hence, $\sec x = -\frac{\sqrt{5}}{2}, \quad \cos x = -\frac{2}{\sqrt{5}}$, $\sin x = \tan x \cos x = -\frac{1}{\sqrt{5}}$.

31. $c = 2, \quad B = \frac{\pi}{3}$

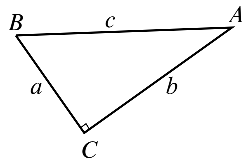
$a = c \cos B = 2 \times \frac{1}{2} = 1$

$b = c \sin B = 2 \times \frac{\sqrt{3}}{2} = \sqrt{3}$

32. $b = 2, \quad B = \frac{\pi}{3}$

$\frac{2}{a} = \tan B = \sqrt{3} \Rightarrow a = \frac{2}{\sqrt{3}}$

$\frac{2}{c} = \sin B = \frac{\sqrt{3}}{2} \Rightarrow c = \frac{4}{\sqrt{3}}$



33. $a = 5, \quad B = \frac{\pi}{6}$

$b = a \tan B = 5 \times \frac{1}{\sqrt{3}} = \frac{5}{\sqrt{3}}$

$c = \sqrt{a^2 + b^2} = \sqrt{25 + \frac{25}{3}} = \frac{10}{\sqrt{3}}$

34. $\sin A = \frac{a}{c} \Rightarrow a = c \sin A$

35. $\frac{a}{b} = \tan A \Rightarrow a = b \tan A$

36. $\cos B = \frac{a}{c} \Rightarrow a = c \cos B$

37. $\frac{b}{a} = \tan B \Rightarrow a = b \cot B$

38. $\sin A = \frac{a}{c} \Rightarrow c = \frac{a}{\sin A}$

39. $\frac{b}{c} = \cos A \Rightarrow c = b \sec A$

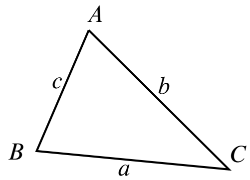
40. $\sin A = \dfrac{a}{c}$

41. $\sin A = \dfrac{a}{c} = \dfrac{\sqrt{c^2 - b^2}}{c}$

42. $\sin A = \dfrac{a}{c} = \dfrac{a}{\sqrt{a^2 + b^2}}$

43. $a = 4,\ b = 3,\ A = \dfrac{\pi}{4}$
$\sin B = b\dfrac{\sin A}{a} = \dfrac{3}{4}\dfrac{1}{\sqrt{2}} = \dfrac{3}{4\sqrt{2}}$

44. Given that $a = 2, b = 2, c = 3$.
Since $a^2 = b^2 + c^2 - 2bc\cos A$,
$\cos A = \dfrac{a^2 - b^2 - c^2}{-2bc}$
$= \dfrac{4 - 4 - 9}{-2(2)(3)} = \dfrac{3}{4}$.



45. $a = 2,\ b = 3,\ c = 4$
$b^2 = a^2 + c^2 - 2ac\cos B$
Thus $\cos B = \dfrac{4 + 16 - 9}{2 \times 2 \times 4} = \dfrac{11}{16}$
$\sin B = \sqrt{1 - \dfrac{11^2}{16^2}} = \dfrac{\sqrt{256 - 121}}{16} = \dfrac{\sqrt{135}}{16}$

46. Given that $a = 2,\ b = 3,\ C = \dfrac{\pi}{4}$.
$c^2 = a^2 + b^2 - 2ab\cos C = 4 + 9 - 2(2)(3)\cos\dfrac{\pi}{4} = 13 - \dfrac{12}{\sqrt{2}}$.
Hence, $c = \sqrt{13 - \dfrac{12}{\sqrt{2}}} \approx 2.12479$.

47. $c = 3,\ A = \dfrac{\pi}{4},\ B = \dfrac{\pi}{3}$ implies $C = \dfrac{5\pi}{12}$
$\dfrac{a}{\sin A} = \dfrac{c}{\sin C} \Rightarrow a = \dfrac{1}{\sqrt{2}}\dfrac{3}{\sin\left(\dfrac{5\pi}{12}\right)}$

$$a = \frac{3}{\sqrt{2}}\frac{1}{\sin\left(\dfrac{7\pi}{12}\right)}$$
$$= \frac{3}{\sqrt{2}}\frac{2\sqrt{2}}{1 + \sqrt{3}} \text{ (by \#5)}$$
$$= \frac{6}{1 + \sqrt{3}}$$

48. Given that $a = 2,\ b = 3,\ C = 35°$. Then
$c^2 = 4 + 9 - 2(2)(3)\cos 35°$, hence $c \approx 1.78050$.

49. $a = 4,\ B = 40°,\ C = 70°$
Thus $A = 70°$.
$\dfrac{b}{\sin 40°} = \dfrac{4}{\sin 70°}$ so $b = 4\dfrac{\sin 40°}{\sin 70°} = 2.736$

50. If $a = 1, b = \sqrt{2}, A = 30°$, then $\dfrac{\sin B}{b} = \dfrac{\sin A}{a} = \dfrac{1}{2}$.
Thus $\sin B = \dfrac{\sqrt{2}}{2} = \dfrac{1}{\sqrt{2}},\ B = \dfrac{\pi}{4}$ or $\dfrac{3\pi}{4}$, and
$C = \pi - \left(\dfrac{\pi}{4} + \dfrac{\pi}{6}\right) = \dfrac{7\pi}{12}$ or $C = \pi - \left(\dfrac{3\pi}{4} + \dfrac{\pi}{6}\right) = \dfrac{\pi}{12}$.
Thus, $\cos C = \cos\dfrac{7\pi}{12} = \cos\left(\dfrac{\pi}{4} + \dfrac{\pi}{3}\right) = \dfrac{1 - \sqrt{3}}{2\sqrt{2}}$ or
$\cos C = \cos\dfrac{\pi}{12} = \cos\left(\dfrac{\pi}{3} - \dfrac{\pi}{4}\right) = \dfrac{1 + \sqrt{3}}{2\sqrt{2}}$.

Hence,

$$\begin{aligned} c^2 &= a^2 + b^2 - 2ab\cos C \\ &= 1 + 2 - 2\sqrt{2}\cos C \\ &= 3 - (1 - \sqrt{3}) \text{ or } 3 - (1 + \sqrt{3}) \\ &= 2 + \sqrt{3} \text{ or } 2 - \sqrt{3}. \end{aligned}$$

Hence, $c = \sqrt{2 + \sqrt{3}}$ or $\sqrt{2 - \sqrt{3}}$.

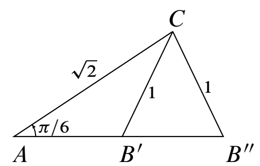


Fig. P.7-50

51. Let h be the height of the pole and x be the distance from C to the base of the pole.
Then $h = x\tan 50°$ and $h = (x + 10)\tan 35°$
Thus $x\tan 50° = x\tan 35° + 10\tan 35°$ so

$$x = \frac{10\tan 35°}{\tan 50° - \tan 35°}$$
$$h = \frac{10\tan 50°\tan 35°}{\tan 50° - \tan 35°} \approx 16.98$$

The pole is about 16.98 metres high.

52. See the following diagram. Since $\tan 40° = h/a$, therefore $a = h/\tan 40°$. Similarly, $b = h/\tan 70°$.
Since $a + b = 2$ km, therefore,

$$\frac{h}{\tan 40°} + \frac{h}{\tan 70°} = 2$$
$$h = \frac{2(\tan 40°\tan 70°)}{\tan 70° + \tan 40°} \approx 1.286 \text{ km.}$$

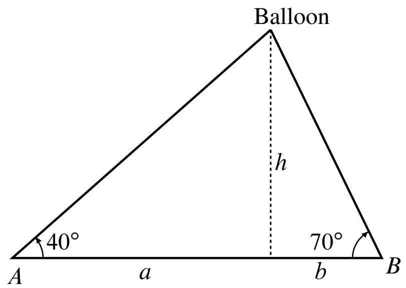


Fig. P.7-52

53. Area $\triangle ABC = \frac{1}{2}|BC|h = \frac{ah}{2} = \frac{ac\sin B}{2} = \frac{ab\sin C}{2}$

By symmetry, area $\triangle ABC$ also $= \frac{1}{2}bc\sin A$

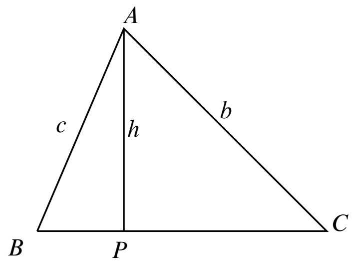


Fig. P.7-53

54. From Exercise 53, area $= \frac{1}{2}ac\sin B$. By Cosine Law, $\cos B = \dfrac{a^2+c^2-b^2}{2ac}$. Thus,

$$\begin{aligned}\sin B &= \sqrt{1-\left(\frac{a^2+c^2-b^2}{2ac}\right)^2}\\ &= \frac{\sqrt{-a^4-b^4-c^4+2a^2b^2+2b^2c^2+2a^2c^2}}{2ac}.\end{aligned}$$

Hence, Area $= \dfrac{\sqrt{-a^4-b^4-c^4+2a^2b^2+2b^2c^2+2a^2c^2}}{4}$ square units. Since,

$$\begin{aligned}&s(s-a)(s-b)(s-c)\\ &\quad= \frac{b+c+a}{2}\,\frac{b+c-a}{2}\,\frac{a-b+c}{2}\,\frac{a+b-c}{2}\\ &\quad= \frac{1}{16}\Big((b+c)^2-a^2\Big)\Big(a^2-(b-c)^2\Big)\\ &\quad= \frac{1}{16}\Big(a^2\big((b+c)^2+(b-c)^2\big)-a^4-(b^2-c^2)^2\Big)\\ &\quad= \frac{1}{16}\Big(2a^2b^2+2a^2c^2-a^4-b^4-c^4+2b^2c^2\Big)\end{aligned}$$

Thus $\sqrt{s(s-a)(s-b)(s-c)}$ = Area of triangle.

CHAPTER 1. LIMITS AND CONTINUITY

Section 1.1 Examples of Velocity, Growth Rate, and Area (page 63)

1. Average velocity $= \dfrac{\Delta x}{\Delta t} = \dfrac{(t+h)^2 - t^2}{h}$ m/s.

2.

h	Avg. vel. over $[2, 2+h]$
1	5.0000
0.1	4.1000
0.01	4.0100
0.001	4.0010
0.0001	4.0001

3. Guess velocity is $v = 4$ m/s at $t = 2$ s.

4. Average velocity on $[2, 2+h]$ is

$$\frac{(2+h)^2 - 4}{(2+h) - 2} = \frac{4 + 4h + h^2 - 4}{h} = \frac{4h + h^2}{h} = 4 + h.$$

As h approaches 0 this average velocity approaches 4 m/s

5. $x = 3t^2 - 12t + 1$ m at time t s.
Average velocity over interval $[1, 2]$ is
$$\frac{(3 \times 2^2 - 12 \times 2 + 1) - (3 \times 1^2 - 12 \times 1 + 1)}{2 - 1} = -3 \text{ m/s}.$$
Average velocity over interval $[2, 3]$ is
$$\frac{(3 \times 3^2 - 12 \times 3 + 1) - (3 \times 2^2 - 12 \times 2 + 1)}{3 - 2} = 3 \text{ m/s}.$$
Average velocity over interval $[1, 3]$ is
$$\frac{(3 \times 3^2 - 12 \times 3 + 1) - (3 \times 1^2 - 12 \times 1 + 1)}{3 - 1} = 0 \text{ m/s}.$$

6. Average velocity over $[t, t+h]$ is

$$\frac{3(t+h)^2 - 12(t+h) + 1 - (3t^2 - 12t + 1)}{(t+h) - t}$$
$$= \frac{6th + 3h^2 - 12h}{h} = 6t + 3h - 12 \text{ m/s}.$$

This average velocity approaches $6t - 12$ m/s as h approaches 0.
At $t = 1$ the velocity is $6 \times 1 - 12 = -6$ m/s.
At $t = 2$ the velocity is $6 \times 2 - 12 = 0$ m/s.
At $t = 3$ the velocity is $6 \times 3 - 12 = 6$ m/s.

7. At $t = 1$ the velocity is $v = -6 < 0$ so the particle is moving to the left.
At $t = 2$ the velocity is $v = 0$ so the particle is stationary.
At $t = 3$ the velocity is $v = 6 > 0$ so the particle is moving to the right.

8. Average velocity over $[t-k, t+k]$ is

$$\frac{3(t+k)^2 - 12(t+k) + 1 - [3(t-k)^2 - 12(t-k) + 1]}{(t+k) - (t-k)}$$
$$= \frac{1}{2k}\Big(3t^2 + 6tk + 3k^2 - 12t - 12k + 1 - 3t^2 + 6tk - 3k^2$$
$$+ 12t - 12k + 1\Big)$$
$$= \frac{12tk - 24k}{2k} = 6t - 12 \text{ m/s},$$

which is the velocity at time t from Exercise 7.

9.

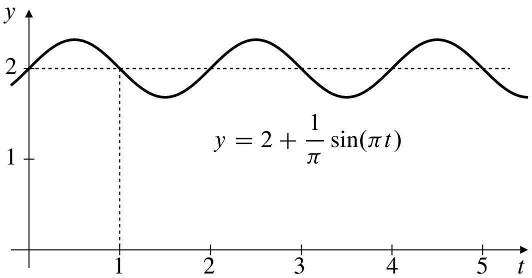


Fig. 1.1-9

At $t = 1$ the height is $y = 2$ ft and the weight is moving downward.

10. Average velocity over $[1, 1+h]$ is

$$\frac{2 + \frac{1}{\pi}\sin\pi(1+h) - \left(2 + \frac{1}{\pi}\sin\pi\right)}{h}$$
$$= \frac{\sin(\pi + \pi h)}{\pi h} = \frac{\sin\pi\cos(\pi h) + \cos\pi\sin(\pi h)}{\pi h}$$
$$= -\frac{\sin(\pi h)}{\pi h}.$$

h	Avg. vel. on $[1, 1+h]$
1.0000	0
0.1000	-0.983631643
0.0100	-0.999835515
0.0010	-0.999998355

11. The velocity at $t = 1$ is about $v = -1$ ft/s. The "–" indicates that the weight is moving downward.

12. We sketched a tangent line to the graph on page 55 in the text at $t = 20$. The line appeared to pass through the points $(10, 0)$ and $(50, 1)$. On day 20 the biomass is growing at about $(1 - 0)/(50 - 10) = 0.025$ mm^2/d.

13. The curve is steepest, and therefore the biomass is growing most rapidly, at about day 45.

14. a)

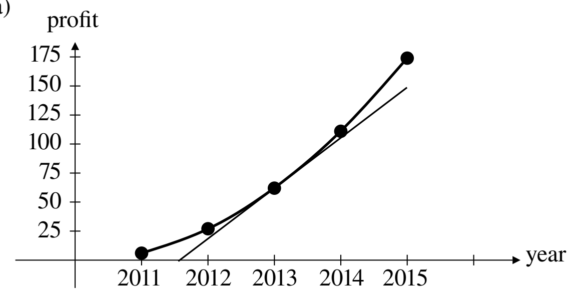


Fig. 1.1-14

b) Average rate of increase in profits between 2010 and 2012 is
$$\frac{174-62}{2012-2010} = \frac{112}{2} = 56 \text{ (thousand\$/yr).}$$

c) Drawing a tangent line to the graph in (a) at $t = 2010$ and measuring its slope, we find that the rate of increase of profits in 2010 is about 43 thousand\$/year.

Section 1.2 Limits of Functions (page 71)

1. From inspecting the graph

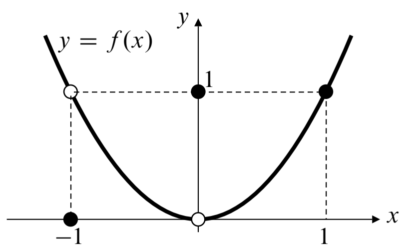


Fig. 1.2-1

we see that
$$\lim_{x\to -1} f(x) = 1, \quad \lim_{x\to 0} f(x) = 0, \quad \lim_{x\to 1} f(x) = 1.$$

2. From inspecting the graph

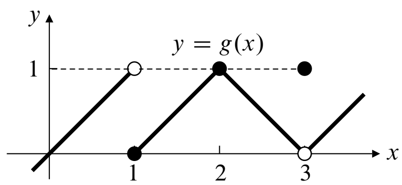


Fig. 1.2-2

we see that
$$\lim_{x\to 1} g(x) \text{ does not exist}$$
(left limit is 1, right limit is 0)
$$\lim_{x\to 2} g(x) = 1, \qquad \lim_{x\to 3} g(x) = 0.$$

3. $\displaystyle\lim_{x\to 1-} g(x) = 1$

4. $\displaystyle\lim_{x\to 1+} g(x) = 0$

5. $\displaystyle\lim_{x\to 3+} g(x) = 0$

6. $\displaystyle\lim_{x\to 3-} g(x) = 0$

7. $\displaystyle\lim_{x\to 4}(x^2-4x+1) = 4^2-4(4)+1 = 1$

8. $\displaystyle\lim_{x\to 2} 3(1-x)(2-x) = 3(-1)(2-2) = 0$

9. $\displaystyle\lim_{x\to 3}\frac{x+3}{x+6} = \frac{3+3}{3+6} = \frac{2}{3}$

10. $\displaystyle\lim_{t\to -4}\frac{t^2}{4-t} = \frac{(-4)^2}{4+4} = 2$

11. $\displaystyle\lim_{x\to 1}\frac{x^2-1}{x+1} = \frac{1^2-1}{1+1} = \frac{0}{2} = 0$

12. $\displaystyle\lim_{x\to -1}\frac{x^2-1}{x+1} = \lim_{x\to -1}(x-1) = -2$

13. $\displaystyle\lim_{x\to 3}\frac{x^2-6x+9}{x^2-9} = \lim_{x\to 3}\frac{(x-3)^2}{(x-3)(x+3)}$
$\displaystyle = \lim_{x\to 3}\frac{x-3}{x+3} = \frac{0}{6} = 0$

14. $\displaystyle\lim_{x\to -2}\frac{x^2+2x}{x^2-4} = \lim_{x\to -2}\frac{x}{x-2} = \frac{-2}{-4} = \frac{1}{2}$

15. $\displaystyle\lim_{h\to 2}\frac{1}{4-h^2}$ does not exist; denominator approaches 0 but numerator does not approach 0.

16. $\displaystyle\lim_{h\to 0}\frac{3h+4h^2}{h^2-h^3} = \lim_{h\to 0}\frac{3+4h}{h-h^2}$ does not exist; denominator approaches 0 but numerator does not approach 0.

17. $\displaystyle\lim_{x\to 9}\frac{\sqrt{x}-3}{x-9} = \lim_{x\to 9}\frac{(\sqrt{x}-3)(\sqrt{x}+3)}{(x-9)(\sqrt{x}+3)}$
$\displaystyle = \lim_{x\to 9}\frac{x-9}{(x-9)(\sqrt{x}+3)} = \lim_{x\to 9}\frac{1}{\sqrt{x}+3} = \frac{1}{6}$

18. $\displaystyle\lim_{h\to 0}\frac{\sqrt{4+h}-2}{h}$
$\displaystyle = \lim_{h\to 0}\frac{4+h-4}{h(\sqrt{4+h}+2)}$
$\displaystyle = \lim_{h\to 0}\frac{1}{\sqrt{4+h}+2} = \frac{1}{4}$

19. $\displaystyle\lim_{x\to\pi}\frac{(x-\pi)^2}{\pi x} = \frac{0^2}{\pi^2} = 0$

20. $\displaystyle\lim_{x\to -2}|x-2| = |-4| = 4$

21. $\displaystyle\lim_{x\to 0}\frac{|x-2|}{x-2} = \frac{|-2|}{-2} = -1$

22. $\lim_{x\to 2}\frac{|x-2|}{x-2} = \lim_{x\to 2}\begin{cases}1, & \text{if } x>2\\ -1, & \text{if } x<2.\end{cases}$
Hence, $\lim_{x\to 2}\frac{|x-2|}{x-2}$ does not exist.

23. $\lim_{t\to 1}\frac{t^2-1}{t^2-2t+1}$
$\lim_{t\to 1}\frac{(t-1)(t+1)}{(t-1)^2} = \lim_{t\to 1}\frac{t+1}{t-1}$ does not exist
(denominator $\to 0$, numerator $\to 2$.)

24. $\lim_{x\to 2}\frac{\sqrt{4-4x+x^2}}{x-2}$
$= \lim_{x\to 2}\frac{|x-2|}{x-2}$ does not exist.

25. $\lim_{t\to 0}\frac{t}{\sqrt{4+t}-\sqrt{4-t}} = \lim_{t\to 0}\frac{t(\sqrt{4+t}+\sqrt{4-t})}{(4+t)-(4-t)}$
$= \lim_{t\to 0}\frac{\sqrt{4+t}+\sqrt{4-t}}{2} = 2$

26. $\lim_{x\to 1}\frac{x^2-1}{\sqrt{x+3}-2} = \lim_{x\to 1}\frac{(x-1)(x+1)(\sqrt{x+3}+2)}{(x+3)-4}$
$= \lim_{x\to 1}(x+1)(\sqrt{x+3}+2) = (2)(\sqrt{4}+2) = 8$

27. $\lim_{t\to 0}\frac{t^2+3t}{(t+2)^2-(t-2)^2}$
$= \lim_{t\to 0}\frac{t(t+3)}{t^2+4t+4-(t^2-4t+4)}$
$= \lim_{t\to 0}\frac{t+3}{8} = \frac{3}{8}$

28. $\lim_{s\to 0}\frac{(s+1)^2-(s-1)^2}{s} = \lim_{s\to 0}\frac{4s}{s} = 4$

29. $\lim_{y\to 1}\frac{y-4\sqrt{y}+3}{y^2-1}$
$= \lim_{y\to 1}\frac{(\sqrt{y}-1)(\sqrt{y}-3)}{(\sqrt{y}-1)(\sqrt{y}+1)(y+1)} = \frac{-2}{4} = \frac{-1}{2}$

30. $\lim_{x\to -1}\frac{x^3+1}{x+1}$
$= \lim_{x\to -1}\frac{(x+1)(x^2-x+1)}{x+1} = 3$

31. $\lim_{x\to 2}\frac{x^4-16}{x^3-8}$
$= \lim_{x\to 2}\frac{(x-2)(x+2)(x^2+4)}{(x-2)(x^2+2x+4)}$
$= \frac{(4)(8)}{4+4+4} = \frac{8}{3}$

32. $\lim_{x\to 8}\frac{x^{2/3}-4}{x^{1/3}-2}$
$= \lim_{x\to 8}\frac{(x^{1/3}-2)(x^{1/3}+2)}{(x^{1/3}-2)}$
$= \lim_{x\to 8}(x^{1/3}+2) = 4$

33. $\lim_{x\to 2}\left(\frac{1}{x-2}-\frac{4}{x^2-4}\right)$
$= \lim_{x\to 2}\frac{x+2-4}{(x-2)(x+2)} = \lim_{x\to 2}\frac{1}{x+2} = \frac{1}{4}$

34. $\lim_{x\to 2}\left(\frac{1}{x-2}-\frac{1}{x^2-4}\right)$
$= \lim_{x\to 2}\frac{x+2-1}{(x-2)(x+2)}$
$= \lim_{x\to 2}\frac{x+1}{(x-2)(x+2)}$ does not exist.

35. $\lim_{x\to 0}\frac{\sqrt{2+x^2}-\sqrt{2-x^2}}{x^2}$
$= \lim_{x\to 0}\frac{(2+x^2)-(2-x^2)}{x^2(\sqrt{2+x^2}+\sqrt{2-x^2})}$
$= \lim_{x\to 0}\frac{2x^2}{x^2\left(\sqrt{2+x^2})+\sqrt{2-x^2}\right)}$
$= \frac{2}{\sqrt{2}+\sqrt{2}} = \frac{1}{\sqrt{2}}$

36. $\lim_{x\to 0}\frac{|3x-1|-|3x+1|}{x}$
$= \lim_{x\to 0}\frac{(3x-1)^2-(3x+1)^2}{x\,(|3x-1|+|3x+1|)}$
$= \lim_{x\to 0}\frac{-12x}{x\,(|3x-1|+|3x+1|)} = \frac{-12}{1+1} = -6$

37. $f(x) = x^2$
$\lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{(x+h)^2-x^2}{h}$
$= \lim_{h\to 0}\frac{2hx+h^2}{h} = \lim_{h\to 0}2x+h = 2x$

38. $f(x) = x^3$
$\lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{(x+h)^3-x^3}{h}$
$= \lim_{h\to 0}\frac{3x^2h+3xh^2+h^3}{h}$
$= \lim_{h\to 0}3x^2+3xh+h^2 = 3x^2$

39. $f(x) = 1/x$
$\lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{\frac{1}{x+h}-\frac{1}{x}}{h}$
$= \lim_{h\to 0}\frac{x-(x+h)}{h(x+h)x}$
$= \lim_{h\to 0}-\frac{1}{(x+h)x} = -\frac{1}{x^2}$

40. $f(x) = 1/x^2$

$$\lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{\dfrac{1}{(x+h)^2}-\dfrac{1}{x^2}}{h} = \lim_{h\to 0}\frac{x^2-(x^2+2xh+h^2)}{h(x+h)^2x^2} = \lim_{h\to 0} -\frac{2x+h}{(x+h)^2x^2} = -\frac{2x}{x^4} = -\frac{2}{x^3}$$

41. $f(x) = \sqrt{x}$

$$\lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{\sqrt{x+h}-\sqrt{x}}{h} = \lim_{h\to 0}\frac{x+h-x}{h(\sqrt{x+h}+\sqrt{x})} = \lim_{h\to 0}\frac{1}{\sqrt{x+h}+\sqrt{x}} = \frac{1}{2\sqrt{x}}$$

42. $f(x) = 1/\sqrt{x}$

$$\lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{\dfrac{1}{\sqrt{x+h}}-\dfrac{1}{\sqrt{x}}}{h} = \lim_{h\to 0}\frac{\sqrt{x}-\sqrt{x+h}}{h\sqrt{x}\sqrt{x+h}} = \lim_{h\to 0}\frac{x-(x+h)}{h\sqrt{x}\sqrt{x+h}(\sqrt{x}+\sqrt{x+h})} = \lim_{h\to 0}\frac{-1}{\sqrt{x}\sqrt{x+h}(\sqrt{x}+\sqrt{x+h})} = \frac{-1}{2x^{3/2}}$$

43. $\lim_{x\to\pi/2}\sin x = \sin\pi/2 = 1$

44. $\lim_{x\to\pi/4}\cos x = \cos\pi/4 = 1/\sqrt{2}$

45. $\lim_{x\to\pi/3}\cos x = \cos\pi/3 = 1/2$

46. $\lim_{x\to 2\pi/3}\sin x = \sin 2\pi/3 = \sqrt{3}/2$

47.

x	$(\sin x)/x$
± 1.0	0.84147098
± 0.1	0.99833417
± 0.01	0.99998333
± 0.001	0.99999983
0.0001	1.00000000

It appears that $\lim_{x\to 0}\dfrac{\sin x}{x} = 1$.

48.

x	$(1-\cos x)/x^2$
± 1.0	0.45969769
± 0.1	0.49958347
± 0.01	0.49999583
± 0.001	0.49999996
0.0001	0.50000000

It appears that $\lim_{x\to 0}\dfrac{1-\cos x}{x^2} = \dfrac{1}{2}$.

49. $\lim_{x\to 2-}\sqrt{2-x} = 0$

50. $\lim_{x\to 2+}\sqrt{2-x}$ does not exist.

51. $\lim_{x\to -2-}\sqrt{2-x} = 2$

52. $\lim_{x\to -2+}\sqrt{2-x} = 2$

53. $\lim_{x\to 0}\sqrt{x^3-x}$ does not exist.
$(x^3 - x < 0 \text{ if } 0 < x < 1)$

54. $\lim_{x\to 0-}\sqrt{x^3-x} = 0$

55. $\lim_{x\to 0+}\sqrt{x^3-x}$ does not exist. (See # 9.)

56. $\lim_{x\to 0+}\sqrt{x^2-x^4} = 0$

57. $\lim_{x\to a-}\dfrac{|x-a|}{x^2-a^2}$
$= \lim_{x\to a-}\dfrac{|x-a|}{(x-a)(x+a)} = -\dfrac{1}{2a} \qquad (a \neq 0)$

58. $\lim_{x\to a+}\dfrac{|x-a|}{x^2-a^2} = \lim_{x\to a+}\dfrac{x-a}{x^2-a^2} = \dfrac{1}{2a}$

59. $\lim_{x\to 2-}\dfrac{x^2-4}{|x+2|} = \dfrac{0}{4} = 0$

60. $\lim_{x\to 2+}\dfrac{x^2-4}{|x+2|} = \dfrac{0}{4} = 0$

61. $f(x) = \begin{cases} x-1 & \text{if } x \le -1 \\ x^2+1 & \text{if } -1 < x \le 0 \\ (x+\pi)^2 & \text{if } x > 0 \end{cases}$
$\lim_{x\to -1-} f(x) = \lim_{x\to -1-} x - 1 = -1 - 1 = -2$

62. $\lim_{x\to -1+} f(x) = \lim_{x\to -1+} x^2 + 1 = 1 + 1 = 2$

63. $\lim_{x\to 0+} f(x) = \lim_{x\to 0+}(x+\pi)^2 = \pi^2$

64. $\lim_{x\to 0-} f(x) = \lim_{x\to 0-} x^2 + 1 = 1$

65. If $\lim_{x\to 4} f(x) = 2$ and $\lim_{x\to 4} g(x) = -3$, then

a) $\lim_{x\to 4}\Big(g(x)+3\Big) = -3 + 3 = 0$

b) $\lim_{x\to 4} x f(x) = 4 \times 2 = 8$

c) $\lim_{x\to 4}\Big(g(x)\Big)^2 = (-3)^2 = 9$

d) $\lim_{x\to 4}\dfrac{g(x)}{f(x)-1} = \dfrac{-3}{2-1} = -3$

66. If $\lim x \to a\, f(x) = 4$ and $\lim_{x\to a} g(x) = -2$, then

a) $\lim_{x\to a}\Big(f(x)+g(x)\Big) = 4 + (-2) = 2$

b) $\lim_{x\to a} f(x)\cdot g(x) = 4 \times (-2) = -8$

c) $\lim_{x\to a} 4g(x) = 4(-2) = -8$

d) $\lim_{x\to a}\dfrac{f(x)}{g(x)} = \dfrac{4}{-2} = -2$

67. If $\lim_{x\to 2}\dfrac{f(x)-5}{x-2} = 3$, then

$$\lim_{x\to 2}\Big(f(x)-5\Big) = \lim_{x\to 2}\frac{f(x)-5}{x-2}(x-2) = 3(2-2) = 0.$$

Thus $\lim_{x\to 2} f(x) = 5$.

68. If $\lim_{x\to 0}\dfrac{f(x)}{x^2} = -2$ then

$\lim_{x\to 0} f(x) = \lim_{x\to 0} x^2\dfrac{f(x)}{x^2} = 0 \times (-2) = 0$,

and similarly, $\lim_{x\to 0}\dfrac{f(x)}{x} = \lim_{x\to 0} x\,\dfrac{f(x)}{x^2} = 0\times(-2) = 0$.

69.

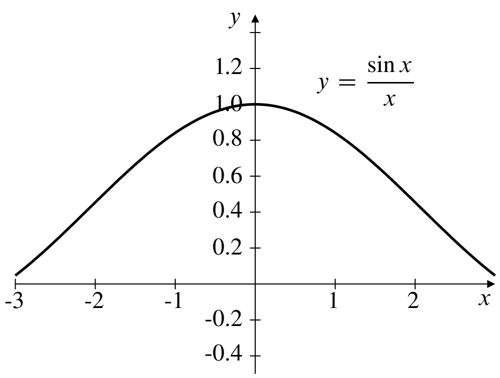


Fig. 1.2-69

$\lim_{x\to 0}\dfrac{\sin x}{x} = 1$

70.

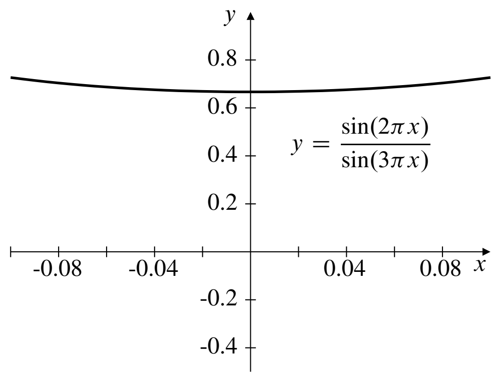


Fig. 1.2-70

$\lim_{x\to 0}\sin(2\pi x)/\sin(3\pi x) = 2/3$

71.

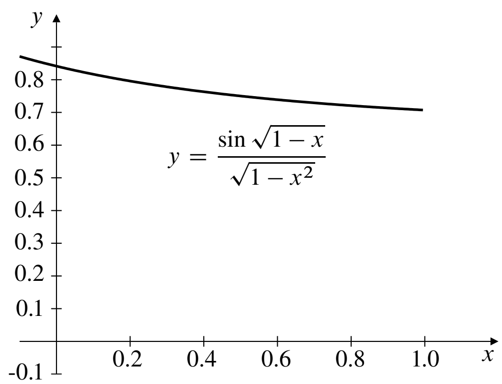


Fig. 1.2-71

$\lim_{x\to 1-}\dfrac{\sin\sqrt{1-x}}{\sqrt{1-x^2}} \approx 0.7071$

72.

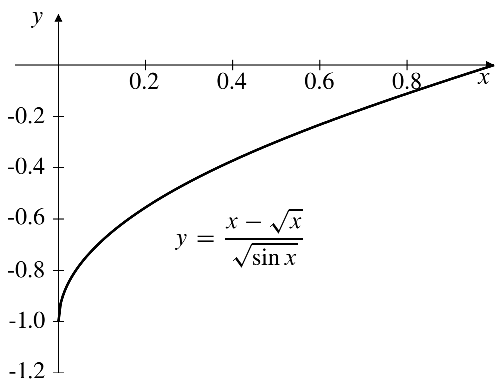


Fig. 1.2-72

$\lim_{x\to 0+}\dfrac{x-\sqrt{x}}{\sqrt{\sin x}} = -1$

73.

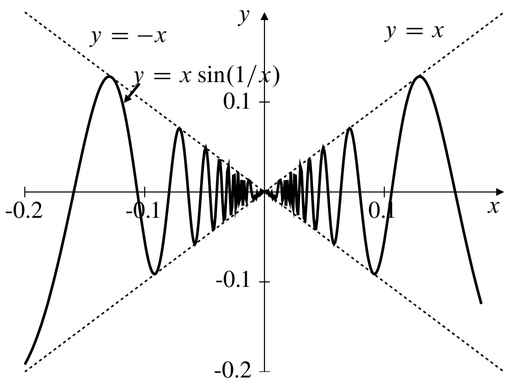


Fig. 1.2-73

$f(x) = x\sin(1/x)$ oscillates infinitely often as x approaches 0, but the amplitude of the oscillations decreases and, in fact, $\lim_{x\to 0} f(x) = 0$. This is predictable because $|x\sin(1/x)| \le |x|$. (See Exercise 95 below.)

74. Since $\sqrt{5-2x^2} \le f(x) \le \sqrt{5-x^2}$ for $-1 \le x \le 1$, and $\lim_{x\to 0}\sqrt{5-2x^2} = \lim_{x\to 0}\sqrt{5-x^2} = \sqrt{5}$, we have $\lim_{x\to 0} f(x) = \sqrt{5}$ by the squeeze theorem.

75. Since $2 - x^2 \le g(x) \le 2\cos x$ for all x, and since $\lim_{x\to 0}(2-x^2) = \lim_{x\to 0} 2\cos x = 2$, we have $\lim_{x\to 0} g(x) = 2$ by the squeeze theorem.

76. a)

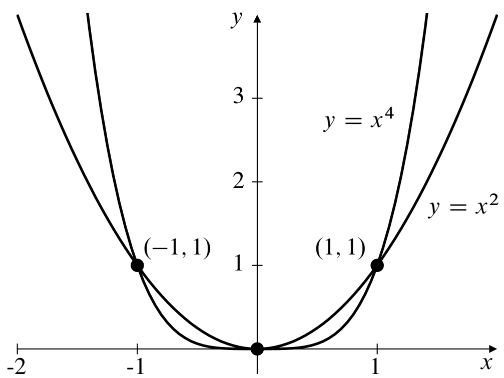


Fig. 1.2-76

b) Since the graph of f lies between those of x^2 and x^4, and since these latter graphs come together at $(\pm 1, 1)$ and at $(0,0)$, we have $\lim_{x\to\pm 1} f(x) = 1$ and $\lim_{x\to 0} f(x) = 0$ by the squeeze theorem.

77. $x^{1/3} < x^3$ on $(-1,0)$ and $(1,\infty)$. $x^{1/3} > x^3$ on $(-\infty,-1)$ and $(0,1)$. The graphs of $x^{1/3}$ and x^3 intersect at $(-1,-1)$, $(0,0)$, and $(1,1)$. If the graph of $h(x)$ lies between those of $x^{1/3}$ and x^3, then we can determine $\lim_{x\to a} h(x)$ for $a = -1$, $a = 0$, and $a = 1$ by the squeeze theorem. In fact

$$\lim_{x\to -1} h(x) = -1, \quad \lim_{x\to 0} h(x) = 0, \quad \lim_{x\to 1} h(x) = 1.$$

78. $f(x) = s\sin\dfrac{1}{x}$ is defined for all $x \ne 0$; its domain is $(-\infty,0)\cup(0,\infty)$. Since $|\sin t| \le 1$ for all t, we have $|f(x)| \le |x|$ and $-|x| \le f(x) \le |x|$ for all $x \ne 0$. Since $\lim_{x\to 0} = (-|x|) = 0 = \lim_{x\to 0}|x|$, we have $\lim_{x\to 0} f(x) = 0$ by the squeeze theorem.

79. $|f(x)| \le g(x) \Rightarrow -g(x) \le f(x) \le g(x)$
Since $\lim\limits_{x\to a} g(x) = 0$, therefore $0 \le \lim\limits_{x\to a} f(x) \le 0$.
Hence, $\lim\limits_{x\to a} f(x) = 0$.
If $\lim\limits_{x\to a} g(x) = 3$, then either $-3 \le \lim\limits_{x\to a} f(x) \le 3$ or $\lim_{x\to a} f(x)$ does not exist.

Section 1.3 Limits at Infinity and Infinite Limits (page 78)

1. $$\lim_{x\to\infty}\frac{x}{2x-3} = \lim_{x\to\infty}\frac{1}{2-(3/x)} = \frac{1}{2}$$

2. $$\lim_{x\to\infty}\frac{x}{x^2-4} = \lim_{x\to\infty}\frac{1/x}{1-(4/x^2)} = \frac{0}{1} = 0$$

3. $$\lim_{x\to\infty}\frac{3x^3-5x^2+7}{8+2x-5x^3} = \lim_{x\to\infty}\frac{3-\dfrac{5}{x}+\dfrac{7}{x^3}}{\dfrac{8}{x^3}+\dfrac{2}{x^2}-5} = -\frac{3}{5}$$

4. $$\lim_{x\to-\infty}\frac{x^2-2}{x-x^2} = \lim_{x\to-\infty}\frac{1-\dfrac{2}{x^2}}{\dfrac{1}{x}-1} = \frac{1}{-1} = -1$$

5. $$\lim_{x\to-\infty}\frac{x^2+3}{x^3+2} = \lim_{x\to-\infty}\frac{\dfrac{1}{x}+\dfrac{3}{x^3}}{1+\dfrac{2}{x^3}} = 0$$

6. $$\lim_{x\to\infty}\frac{x^2+\sin x}{x^2+\cos x} = \lim_{x\to\infty}\frac{1+\dfrac{\sin x}{x^2}}{1+\dfrac{\cos x}{x^2}} = \frac{1}{1} = 1$$

We have used the fact that $\lim_{x\to\infty}\dfrac{\sin x}{x^2} = 0$ (and similarly for cosine) because the numerator is bounded while the denominator grows large.

7. $$\lim_{x\to\infty}\frac{3x+2\sqrt{x}}{1-x} = \lim_{x\to\infty}\frac{3+\dfrac{2}{\sqrt{x}}}{\dfrac{1}{x}-1} = -3$$

8. $\lim_{x\to\infty} \frac{2x-1}{\sqrt{3x^2+x+1}}$

$= \lim_{x\to\infty} \frac{x\left(2-\frac{1}{x}\right)}{|x|\sqrt{3+\frac{1}{x}+\frac{1}{x^2}}}$ (but $|x| = x$ as $x \to \infty$)

$= \lim_{x\to\infty} \frac{2-\frac{1}{x}}{\sqrt{3+\frac{1}{x}+\frac{1}{x^2}}} = \frac{2}{\sqrt{3}}$

9. $\lim_{x\to-\infty} \frac{2x-1}{\sqrt{3x^2+x+1}}$

$= \lim_{x\to-\infty} \frac{2-\frac{1}{x}}{-\sqrt{3+\frac{1}{x}+\frac{1}{x^2}}} = -\frac{2}{\sqrt{3}},$

because $x \to -\infty$ implies that $x < 0$ and so $\sqrt{x^2} = -x$.

10. $\lim_{x\to-\infty} \frac{2x-5}{|3x+2|} = \lim_{x\to-\infty} \frac{2x-5}{-(3x+2)} = -\frac{2}{3}$

11. $\lim_{x\to 3} \frac{1}{3-x}$ does not exist.

12. $\lim_{x\to 3} \frac{1}{(3-x)^2} = \infty$

13. $\lim_{x\to 3-} \frac{1}{3-x} = \infty$

14. $\lim_{x\to 3+} \frac{1}{3-x} = -\infty$

15. $\lim_{x\to -5/2} \frac{2x+5}{5x+2} = \frac{0}{\frac{-25}{2}+2} = 0$

16. $\lim_{x\to -2/5} \frac{2x+5}{5x+2}$ does not exist.

17. $\lim_{x\to -(2/5)-} \frac{2x+5}{5x+2} = -\infty$

18. $\lim_{x\to -2/5+} \frac{2x+5}{5x+2} = \infty$

19. $\lim_{x\to 2+} \frac{x}{(2-x)^3} = -\infty$

20. $\lim_{x\to 1-} \frac{x}{\sqrt{1-x^2}} = \infty$

21. $\lim_{x\to 1+} \frac{1}{|x-1|} = \infty$

22. $\lim_{x\to 1-} \frac{1}{|x-1|} = \infty$

23. $\lim_{x\to 2} \frac{x-3}{x^2-4x+4} = \lim_{x\to 2} \frac{x-3}{(x-2)^2} = -\infty$

24. $\lim_{x\to 1+} \frac{\sqrt{x^2-x}}{x-x^2} = \lim_{x\to 1+} \frac{-1}{\sqrt{x^2-x}} = -\infty$

25. $\lim_{x\to\infty} \frac{x+x^3+x^5}{1+x^2+x^3}$

$= \lim_{x\to\infty} \frac{\frac{1}{x^2}+1+x^2}{\frac{1}{x^3}+\frac{1}{x}+1} = \infty$

26. $\lim_{x\to\infty} \frac{x^3+3}{x^2+2} = \lim_{x\to\infty} \frac{x+\frac{3}{x^2}}{1+\frac{2}{x^2}} = \infty$

27. $\lim_{x\to\infty} \frac{x\sqrt{x+1}\left(1-\sqrt{2x+3}\right)}{7-6x+4x^2}$

$= \lim_{x\to\infty} \frac{x^2\left(\sqrt{1+\frac{1}{x}}\right)\left(\frac{1}{\sqrt{x}}-\sqrt{2+\frac{3}{x}}\right)}{x^2\left(\frac{7}{x^2}-\frac{6}{x}+4\right)}$

$= \frac{1(-\sqrt{2})}{4} = -\frac{1}{4}\sqrt{2}$

28. $\lim_{x\to\infty} \left(\frac{x^2}{x+1}-\frac{x^2}{x-1}\right) = \lim_{x\to\infty} \frac{-2x^2}{x^2-1} = -2$

29. $\lim_{x\to-\infty} \left(\sqrt{x^2+2x}-\sqrt{x^2-2x}\right)$

$= \lim_{x\to-\infty} \frac{(x^2+2x)-(x^2-2x)}{\sqrt{x^2+2x}+\sqrt{x^2-2x}}$

$= \lim_{x\to-\infty} \frac{4x}{(-x)\left(\sqrt{1+\frac{2}{x}}+\sqrt{1-\frac{2}{x}}\right)}$

$= -\frac{4}{1+1} = -2$

30. $\lim_{x\to\infty} \left(\sqrt{x^2+2x}-\sqrt{x^2-2x}\right)$

$= \lim_{x\to\infty} \frac{x^2+2x-x^2+2x}{\sqrt{x^2+2x}+\sqrt{x^2-2x}}$

$= \lim_{x\to\infty} \frac{4x}{x\sqrt{1+\frac{2}{x}}+x\sqrt{1-\frac{2}{x}}}$

$= \lim_{x\to\infty} \frac{4}{\sqrt{1+\frac{2}{x}}+\sqrt{1-\frac{2}{x}}} = \frac{4}{2} = 2$

31. $\lim_{x\to\infty} \frac{1}{\sqrt{x^2-2x}-x}$

$= \lim_{x\to\infty} \frac{\sqrt{x^2-2x}+x}{(\sqrt{x^2-2x}+x)(\sqrt{x^2-2x}-x)}$

$= \lim_{x\to\infty} \frac{\sqrt{x^2-2x}+x}{x^2-2x-x^2}$

$= \lim_{x\to\infty} \frac{x(\sqrt{1-(2/x)}+1)}{-2x} = \frac{2}{-2} = -1$

32. $\lim_{x\to-\infty} \frac{1}{\sqrt{x^2+2x}-x} = \lim_{x\to-\infty} \frac{1}{|x|(\sqrt{1+(2/x)}+1} = 0$

33. By Exercise 35, $y = -1$ is a horizontal asymptote (at the right) of $y = \dfrac{1}{\sqrt{x^2 - 2x} - x}$. Since

$$\lim_{x\to-\infty} \frac{1}{\sqrt{x^2 - 2x} - x} = \lim_{x\to-\infty} \frac{1}{|x|(\sqrt{1 - (2/x)} + 1} = 0,$$

$y = 0$ is also a horizontal asymptote (at the left). Now $\sqrt{x^2 - 2x} - x = 0$ if and only if $x^2 - 2x = x^2$, that is, if and only if $x = 0$. The given function is undefined at $x = 0$, and where $x^2 - 2x < 0$, that is, on the interval $[0, 2]$. Its only vertical asymptote is at $x = 0$, where $\lim_{x\to 0-} \dfrac{1}{\sqrt{x^2 - 2x} - x} = \infty$.

34. Since $\displaystyle\lim_{x\to\infty} \frac{2x-5}{|3x+2|} = \frac{2}{3}$ and $\displaystyle\lim_{x\to-\infty} \frac{2x-5}{|3x+2|} = -\frac{2}{3}$, $y = \pm(2/3)$ are horizontal asymptotes of $y = (2x-5)/|3x+2|$. The only vertical asymptote is $x = -2/3$, which makes the denominator zero.

35. $\displaystyle\lim_{x\to 0+} f(x) = 1$

36. $\displaystyle\lim_{x\to 1} f(x) = \infty$

37.

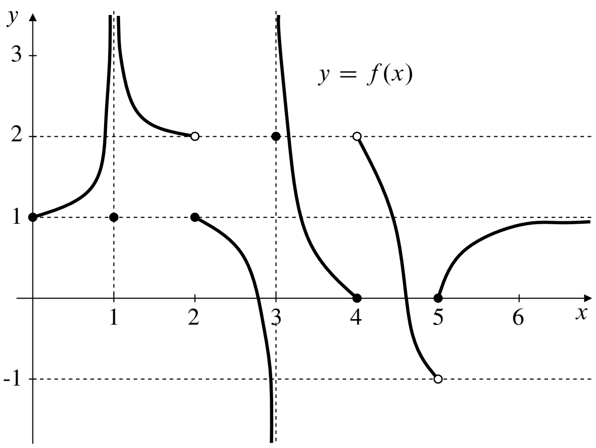


Fig. 1.3-37

$\lim_{x\to 2+} f(x) = 1$

38. $\displaystyle\lim_{x\to 2-} f(x) = 2$

39. $\displaystyle\lim_{x\to 3-} f(x) = -\infty$

40. $\displaystyle\lim_{x\to 3+} f(x) = \infty$

41. $\displaystyle\lim_{x\to 4+} f(x) = 2$

42. $\displaystyle\lim_{x\to 4-} f(x) = 0$

43. $\displaystyle\lim_{x\to 5-} f(x) = -1$

44. $\displaystyle\lim_{x\to 5+} f(x) = 0$

45. $\displaystyle\lim_{x\to\infty} f(x) = 1$

46. horizontal: $y = 1$; vertical: $x = 1$, $x = 3$.

47. $\displaystyle\lim_{x\to 3+} \lfloor x \rfloor = 3$

48. $\displaystyle\lim_{x\to 3-} \lfloor x \rfloor = 2$

49. $\displaystyle\lim_{x\to 3} \lfloor x \rfloor$ does not exist

50. $\displaystyle\lim_{x\to 2.5} \lfloor x \rfloor = 2$

51. $\displaystyle\lim_{x\to 0+} \lfloor 2 - x \rfloor = \lim_{x\to 2-} \lfloor x \rfloor = 1$

52. $\displaystyle\lim_{x\to -3-} \lfloor x \rfloor = -4$

53. $\displaystyle\lim_{t\to t_0} C(t) = C(t_0)$ except at integers t_0
$\displaystyle\lim_{t\to t_0-} C(t) = C(t_0)$ everywhere
$\displaystyle\lim_{t\to t_0+} C(t) = C(t_0)$ if $t_0 \neq$ an integer
$\displaystyle\lim_{t\to t_0+} C(t) = C(t_0) + 1.5$ if t_0 is an integer

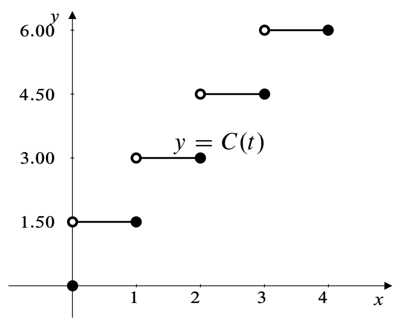


Fig. 1.3-53

54. $\displaystyle\lim_{x\to 0+} f(x) = L$
(a) If f is even, then $f(-x) = f(x)$.
Hence, $\displaystyle\lim_{x\to 0-} f(x) = L$.
(b) If f is odd, then $f(-x) = -f(x)$.
Therefore, $\displaystyle\lim_{x\to 0-} f(x) = -L$.

55. $\displaystyle\lim_{x\to 0+} f(x) = A$, $\displaystyle\lim_{x\to 0-} f(x) = B$

a) $\displaystyle\lim_{x\to 0+} f(x^3 - x) = B$ (since $x^3 - x < 0$ if $0 < x < 1$)

b) $\displaystyle\lim_{x\to 0-} f(x^3 - x) = A$ (because $x^3 - x > 0$ if $-1 < x < 0$)

c) $\displaystyle\lim_{x\to 0-} f(x^2 - x^4) = A$

d) $\displaystyle\lim_{x\to 0+} f(x^2 - x^4) = A$ (since $x^2 - x^4 > 0$ for $0 < |x| < 1$)

Section 1.4 Continuity (page 87)

1. g is continuous at $x = -2$, discontinuous at $x = -1$, $0, 1$, and 2. It is left continuous at $x = 0$ and right continuous at $x = 1$.

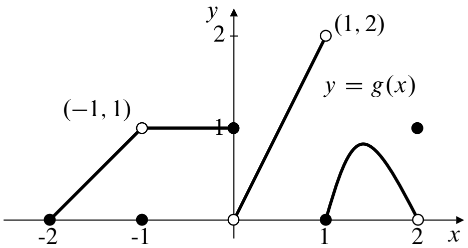


Fig. 1.4-1

2. g has removable discontinuities at $x = -1$ and $x = 2$. Redefine $g(-1) = 1$ and $g(2) = 0$ to make g continuous at those points.

3. g has no absolute maximum value on $[-2, 2]$. It takes on every positive real value less than 2, but does not take the value 2. It has absolute minimum value 0 on that interval, assuming this value at the three points $x = -2$, $x = -1$, and $x = 1$.

4. Function f is discontinuous at $x = 1, 2, 3, 4$, and 5. f is left continuous at $x = 4$ and right continuous at $x = 2$ and $x = 5$.

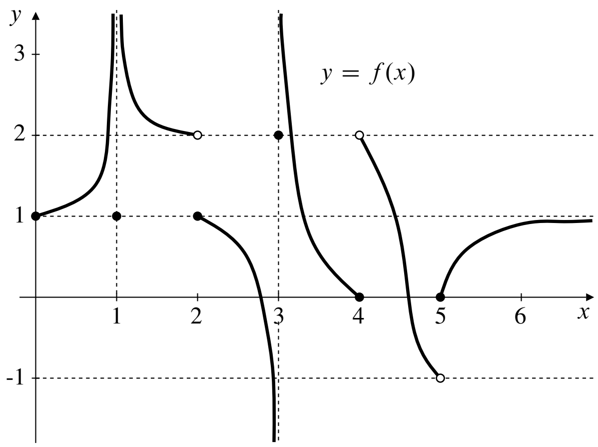


Fig. 1.4-4

5. f cannot be redefined at $x = 1$ to become continuous there because $\lim_{x\to 1} f(x)$ $(= \infty)$ does not exist. (∞ is not a real number.)

6. $\operatorname{sgn} x$ is not defined at $x = 0$, so cannot be either continuous or discontinuous there. (Functions can be continuous or discontinuous only at points in their domains!)

7. $f(x) = \begin{cases} x & \text{if } x < 0 \\ x^2 & \text{if } x \ge 0 \end{cases}$ is continuous everywhere on the real line, even at $x = 0$ where its left and right limits are both 0, which is $f(0)$.

8. $f(x) = \begin{cases} x & \text{if } x < -1 \\ x^2 & \text{if } x \ge -1 \end{cases}$ is continuous everywhere on the real line except at $x = -1$ where it is right continuous, but not left continuous.

$$\lim_{x\to -1-} f(x) = \lim_{x\to -1-} x = -1 \ne 1$$
$$= f(-1) = \lim_{x\to -1+} x^2 = \lim_{x\to -1+} f(x).$$

9. $f(x) = \begin{cases} 1/x^2 & \text{if } x \ne 0 \\ 0 & \text{if } x = 0 \end{cases}$ is continuous everywhere except at $x = 0$, where it is neither left nor right continuous since it does not have a real limit there.

10. $f(x) = \begin{cases} x^2 & \text{if } x \le 1 \\ 0.987 & \text{if } x > 1 \end{cases}$ is continuous everywhere except at $x = 1$, where it is left continuous but not right continuous because $0.987 \ne 1$. Close, as they say, but no cigar.

11. The least integer function $\lceil x \rceil$ is continuous everywhere on $\mathbb{R}$ except at the integers, where it is left continuous but not right continuous.

12. $C(t)$ is discontinuous only at the integers. It is continuous on the left at the integers, but not on the right.

13. Since $\dfrac{x^2 - 4}{x - 2} = x + 2$ for $x \ne 2$, we can define the function to be $2 + 2 = 4$ at $x = 2$ to make it continuous there. The continuous extension is $x + 2$.

14. Since $\dfrac{1 + t^3}{1 - t^2} = \dfrac{(1 + t)(1 - t + t^2)}{(1 + t)(1 - t)} = \dfrac{1 - t + t^2}{1 - t}$ for $t \ne -1$, we can define the function to be $3/2$ at $t = -1$ to make it continuous there. The continuous extension is $\dfrac{1 - t + t^2}{1 - t}$.

15. Since $\dfrac{t^2 - 5t + 6}{t^2 - t - 6} = \dfrac{(t - 2)(t - 3)}{(t + 2)(t - 3)} = \dfrac{t - 2}{t + 2}$ for $t \ne 3$, we can define the function to be $1/5$ at $t = 3$ to make it continuous there. The continuous extension is $\dfrac{t - 2}{t + 2}$.

16. Since
$$\frac{x^2 - 2}{x^4 - 4} = \frac{(x - \sqrt{2})(x + \sqrt{2})}{(x - \sqrt{2})(x + \sqrt{2})(x^2 + 2)} = \frac{x + \sqrt{2}}{(x + \sqrt{2})(x^2 + 2)}$$
for $x \ne \sqrt{2}$, we can define the function to be $1/4$ at $x = \sqrt{2}$ to make it continuous there. The continuous extension is $\dfrac{x + \sqrt{2}}{(x + \sqrt{2})(x^2 + 2)}$. (Note: cancelling the $x + \sqrt{2}$ factors provides a further continuous extension to $x = -\sqrt{2}$.

17. $\lim_{x\to 2+} f(x) = k - 4$ and $\lim_{x\to 2-} f(x) = 4 = f(2)$. Thus f will be continuous at $x = 2$ if $k - 4 = 4$, that is, if $k = 8$.

18. $\lim_{x\to 3-} g(x) = 3 - m$ and $\lim_{x\to 3+} g(x) = 1 - 3m = g(3)$. Thus g will be continuous at $x = 3$ if $3 - m = 1 - 3m$, that is, if $m = -1$.

19. x^2 has no maximum value on $-1 < x < 1$; it takes all positive real values less than 1, but it does not take the value 1. It does have a minimum value, namely 0 taken on at $x = 0$.

20. The Max-Min Theorem says that a continuous function defined on a closed, finite interval must have maximum and minimum values. It does not say that other functions cannot have such values. The Heaviside function is not continuous on $[-1, 1]$ (because it is discontinuous at $x = 0$), but it still has maximum and minimum values. Do not confuse a theorem with its converse.

21. Let the numbers be x and y, where $x \geq 0$, $y \geq 0$, and $x + y = 8$. If P is the product of the numbers, then

$$P = xy = x(8 - x) = 8x - x^2 = 16 - (x - 4)^2.$$

Therefore $P \leq 16$, so P is bounded. Clearly $P = 16$ if $x = y = 4$, so the largest value of P is 16.

22. Let the numbers be x and y, where $x \geq 0$, $y \geq 0$, and $x + y = 8$. If S is the sum of their squares then

$$\begin{aligned} S &= x^2 + y^2 = x^2 + (8 - x)^2 \\ &= 2x^2 - 16x + 64 = 2(x - 4)^2 + 32. \end{aligned}$$

Since $0 \leq x \leq 8$, the maximum value of S occurs at $x = 0$ or $x = 8$, and is 64. The minimum value occurs at $x = 4$ and is 32.

23. Since $T = 100 - 30x + 3x^2 = 3(x - 5)^2 + 25$, T will be minimum when $x = 5$. Five programmers should be assigned, and the project will be completed in 25 days.

24. If x desks are shipped, the shipping cost per desk is

$$\begin{aligned} C = \frac{245x - 30x^2 + x^3}{x} &= x^2 - 30x + 245 \\ &= (x - 15)^2 + 20. \end{aligned}$$

This cost is minimized if $x = 15$. The manufacturer should send 15 desks in each shipment, and the shipping cost will then be \$20 per desk.

25. $f(x) = \dfrac{x^2 - 1}{x} = \dfrac{(x - 1)(x + 1)}{x}$
$f = 0$ at $x = \pm 1$. f is not defined at 0.
$f(x) > 0$ on $(-1, 0)$ and $(1, \infty)$.
$f(x) < 0$ on $(-\infty, -1)$ and $(0, 1)$.

26. $f(x) = x^2 + 4x + 3 = (x + 1)(x + 3)$
$f(x) > 0$ on $(-\infty, -3)$ and $(-1, \infty)$
$f(x) < 0$ on $(-3, -1)$.

27. $f(x) = \dfrac{x^2 - 1}{x^2 - 4} = \dfrac{(x - 1)(x + 1)}{(x - 2)(x + 2)}$
$f = 0$ at $x = \pm 1$.
f is not defined at $x = \pm 2$.
$f(x) > 0$ on $(-\infty, -2)$, $(-1, 1)$, and $(2, \infty)$.
$f(x) < 0$ on $(-2, -1)$ and $(1, 2)$.

28. $f(x) = \dfrac{x^2 + x - 2}{x^3} = \dfrac{(x + 2)(x - 1)}{x^3}$
$f(x) > 0$ on $(-2, 0)$ and $(1, \infty)$
$f(x) < 0$ on $(-\infty, -2)$ and $(0, 1)$.

29. $f(x) = x^3 + x - 1$, $f(0) = -1$, $f(1) = 1$.
Since f is continuous and changes sign between 0 and 1, it must be zero at some point between 0 and 1 by IVT.

30. $f(x) = x^3 - 15x + 1$ is continuous everywhere.
$f(-4) = -3$, $f(-3) = 19$, $f(1) = -13$, $f(4) = 5$.
Because of the sign changes f has a zero between -4 and -3, another zero between -3 and 1, and another between 1 and 4.

31. $F(x) = (x - a)^2(x - b)^2 + x$. Without loss of generality, we can assume that $a < b$. Being a polynomial, F is continuous on $[a, b]$. Also $F(a) = a$ and $F(b) = b$. Since $a < \frac{1}{2}(a + b) < b$, the Intermediate-Value Theorem guarantees that there is an x in (a, b) such that $F(x) = (a + b)/2$.

32. Let $g(x) = f(x) - x$. Since $0 \leq f(x) \leq 1$ if $0 \leq x \leq 1$, therefore, $g(0) \geq 0$ and $g(1) \leq 0$. If $g(0) = 0$ let $c = 0$, or if $g(1) = 0$ let $c = 1$. (In either case $f(c) = c$.) Otherwise, $g(0) > 0$ and $g(1) < 0$, and, by IVT, there exists c in $(0, 1)$ such that $g(c) = 0$, i.e., $f(c) = c$.

33. The domain of an even function is symmetric about the y-axis. Since f is continuous on the right at $x = 0$, therefore it must be defined on an interval $[0, h]$ for some $h > 0$. Being even, f must therefore be defined on $[-h, h]$. If $x = -y$, then

$$\lim_{x \to 0-} f(x) = \lim_{y \to 0+} f(-y) = \lim_{y \to 0+} f(y) = f(0).$$

Thus, f is continuous on the left at $x = 0$. Being continuous on both sides, it is therefore continuous.

34. f odd $\Leftrightarrow f(-x) = -f(x)$
f continuous on the right $\Leftrightarrow \lim_{x \to 0+} f(x) = f(0)$
Therefore, letting $t = -x$, we obtain

$$\begin{aligned} \lim_{x \to 0-} f(x) = \lim_{t \to 0+} f(-t) &= \lim_{t \to 0+} -f(t) \\ &= -f(0) = f(-0) = f(0). \end{aligned}$$

Therefore f is continuous at 0 and $f(0) = 0$.

35. max 1.593 at -0.831, min -0.756 at 0.629

36. max 0.133 at $x = 1.437$; min -0.232 at $x = -1.805$

37. max 10.333 at $x = 3$; min 4.762 at $x = 1.260$

38. max 1.510 at $x = 0.465$; min 0 at $x = 0$ and $x = 1$

39. root $x = 0.682$

40. root $x = 0.739$

41. roots $x = -0.637$ and $x = 1.410$

42. roots $x = -0.7244919590$ and $x = 1.220744085$

43. fsolve gives an approximation to the single real root to 10 significant figures; solve gives the three roots (including a complex conjugate pair) in exact form involving the quantity $\left(108 + 12\sqrt{69}\right)^{1/3}$; evalf(solve) gives approximations to the three roots using 10 significant figures for the real and imaginary parts.

Section 1.5 The Formal Definition of Limit (page 92)

1. We require $39.9 \le L \le 40.1$. Thus

$$\begin{aligned}39.9 &\le 39.6 + 0.025T \le 40.1\\ 0.3 &\le 0.025T \le 0.5\\ 12 &\le T \le 20.\end{aligned}$$

The temperature should be kept between 12 °C and 20 °C.

2. Since 1.2% of 8,000 is 96, we require the edge length x of the cube to satisfy $7904 \le x^3 \le 8096$. It is sufficient that $19.920 \le x \le 20.079$. The edge of the cube must be within 0.079 cm of 20 cm.

3.
$$\begin{aligned}3 - 0.02 &\le 2x - 1 \le 3 + 0.02\\ 3.98 &\le 2x \le 4.02\\ 1.99 &\le x \le 2.01\end{aligned}$$

4.
$$\begin{aligned}4 - 0.1 &\le x^2 \le 4 + 0.1\\ 1.9749 &\le x \le 2.0024\end{aligned}$$

5.
$$\begin{aligned}1 - 0.1 &\le \sqrt{x} \le 1.1\\ 0.81 &\le x \le 1.21\end{aligned}$$

6.
$$\begin{aligned}-2 - 0.01 &\le \frac{1}{x} \le -2 + 0.01\\ -\frac{1}{2.01} &\ge x \ge -\frac{1}{1.99}\\ -0.5025 &\le x \le -0.4975\end{aligned}$$

7. We need $-0.03 \le (3x + 1) - 7 \le 0.03$, which is equivalent to $-0.01 \le x - 2 \le 0.01$ Thus $\delta = 0.01$ will do.

8. We need $-0.01 \le \sqrt{2x + 3} - 3 \le 0.01$. Thus

$$\begin{aligned}2.99 &\le \sqrt{2x + 3} \le 3.01\\ 8.9401 &\le 2x + 3 \le 9.0601\\ 2.97005 &\le x \le 3.03005\\ 3 - 0.02995 &\le x - 3 \le 0.03005.\end{aligned}$$

Here $\delta = 0.02995$ will do.

9. We need $8 - 0.2 \le x^3 \le 8.2$, or $1.9832 \le x \le 2.0165$. Thus, we need $-0.0168 \le x - 2 \le 0.0165$. Here $\delta = 0.0165$ will do.

10. We need $1 - 0.05 \le 1/(x + 1) \le 1 + 0.05$, or $1.0526 \ge x + 1 \ge 0.9524$. This will occur if $-0.0476 \le x \le 0.0526$. In this case we can take $\delta = 0.0476$.

11. To be proved: $\lim_{x\to 1} (3x + 1) = 4$.
Proof: Let $\epsilon > 0$ be given. Then $|(3x + 1) - 4| < \epsilon$ holds if $3|x - 1| < \epsilon$, and so if $|x - 1| < \delta = \epsilon/3$. This confirms the limit.

12. To be proved: $\lim_{x\to 2} (5 - 2x) = 1$.
Proof: Let $\epsilon > 0$ be given. Then $|(5 - 2x) - 1| < \epsilon$ holds if $|2x - 4| < \epsilon$, and so if $|x - 2| < \delta = \epsilon/2$. This confirms the limit.

13. To be proved: $\lim_{x\to 0} x^2 = 0$.
Let $\epsilon > 0$ be given. Then $|x^2 - 0| < \epsilon$ holds if $|x - 0| = |x| < \delta = \sqrt{\epsilon}$.

14. To be proved: $\lim_{x\to 2} \dfrac{x - 2}{1 + x^2} = 0$.
Proof: Let $\epsilon > 0$ be given. Then

$$\left|\frac{x - 2}{1 + x^2} - 0\right| = \frac{|x - 2|}{1 + x^2} \le |x - 2| < \epsilon$$

provided $|x - 2| < \delta = \epsilon$.

15. To be proved: $\lim_{x\to 1/2} \dfrac{1 - 4x^2}{1 - 2x} = 2$.
Proof: Let $\epsilon > 0$ be given. Then if $x \ne 1/2$ we have

$$\left|\frac{1 - 4x^2}{1 - 2x} - 2\right| = |(1 + 2x) - 2| = |2x - 1| = 2\left|x - \frac{1}{2}\right| < \epsilon$$

provided $|x - \frac{1}{2}| < \delta = \epsilon/2$.

16. To be proved: $\lim_{x\to -2} \dfrac{x^2 + 2x}{x + 2} = -2$.
Proof: Let $\epsilon > 0$ be given. For $x \ne -2$ we have

$$\left|\frac{x^2 + 2x}{x + 2} - (-2)\right| = |x + 2| < \epsilon$$

provided $|x + 2| < \delta = \epsilon$. This completes the proof.

17. To be proved: $\lim_{x\to 1} \dfrac{1}{x + 1} = \dfrac{1}{2}$.
Proof: Let $\epsilon > 0$ be given. We have

$$\left|\frac{1}{x + 1} - \frac{1}{2}\right| = \left|\frac{1 - x}{2(x + 1)}\right| = \frac{|x - 1|}{2|x + 1|}.$$

If $|x - 1| < 1$, then $0 < x < 2$ and $1 < x + 1 < 3$, so that $|x + 1| > 1$. Let $\delta = \min(1, 2\epsilon)$. If $|x - 1| < \delta$, then

$$\left|\frac{1}{x + 1} - \frac{1}{2}\right| = \frac{|x - 1|}{2|x + 1|} < \frac{2\epsilon}{2} = \epsilon.$$

This establishes the required limit.

18. To be proved: $\lim_{x\to-1}\dfrac{x+1}{x^2-1}=-\dfrac{1}{2}$.
Proof: Let $\epsilon>0$ be given. If $x\neq-1$, we have

$$\left|\frac{x+1}{x^2-1}-\left(-\frac{1}{2}\right)\right|=\left|\frac{1}{x-1}-\left(-\frac{1}{2}\right)\right|=\frac{|x+1|}{2|x-1|}.$$

If $|x+1|<1$, then $-2<x<0$, so $-3<x-1<-1$ and $|x-1|>1$. Ler $\delta=\min(1,2\epsilon)$. If $0<|x-(-1)|<\delta$ then $|x-1|>1$ and $|x+1|<2\epsilon$. Thus

$$\left|\frac{x+1}{x^2-1}-\left(-\frac{1}{2}\right)\right|=\frac{|x+1|}{2|x-1|}<\frac{2\epsilon}{2}=\epsilon.$$

This completes the required proof.

19. To be proved: $\lim_{x\to1}\sqrt{x}=1$.
Proof: Let $\epsilon>0$ be given. We have

$$|\sqrt{x}-1|=\left|\frac{x-1}{\sqrt{x}+1}\right|\le|x-1|<\epsilon$$

provided $|x-1|<\delta=\epsilon$. This completes the proof.

20. To be proved: $\lim_{x\to2}x^3=8$.
Proof: Let $\epsilon>0$ be given. We have $|x^3-8|=|x-2||x^2+2x+4|$. If $|x-2|<1$, then $1<x<3$ and $x^2<9$. Therefore $|x^2+2x+4|\le9+2\times3+4=19$. If $|x-2|<\delta=\min(1,\epsilon/19)$, then

$$|x^3-8|=|x-2||x^2+2x+4|<\frac{\epsilon}{19}\times19=\epsilon.$$

This completes the proof.

21. We say that $\lim_{x\to a-}f(x)=L$ if the following condition holds: for every number $\epsilon>0$ there exists a number $\delta>0$, depending on ϵ, such that

$$a-\delta<x<a\quad\text{implies}\quad|f(x)-L|<\epsilon.$$

22. We say that $\lim_{x\to-\infty}f(x)=L$ if the following condition holds: for every number $\epsilon>0$ there exists a number $R>0$, depending on ϵ, such that

$$x<-R\quad\text{implies}\quad|f(x)-L|<\epsilon.$$

23. We say that $\lim_{x\to a}f(x)=-\infty$ if the following condition holds: for every number $B>0$ there exists a number $\delta>0$, depending on B, such that

$$0<|x-a|<\delta\quad\text{implies}\quad f(x)<-B.$$

24. We say that $\lim_{x\to\infty}f(x)=\infty$ if the following condition holds: for every number $B>0$ there exists a number $R>0$, depending on B, such that

$$x>R\quad\text{implies}\quad f(x)>B.$$

25. We say that $\lim_{x\to a+}f(x)=-\infty$ if the following condition holds: for every number $B>0$ there exists a number $\delta>0$, depending on R, such that

$$a<x<a+\delta\quad\text{implies}\quad f(x)<-B.$$

26. We say that $\lim_{x\to a-}f(x)=\infty$ if the following condition holds: for every number $B>0$ there exists a number $\delta>0$, depending on B, such that

$$a-\delta<x<a\quad\text{implies}\quad f(x)>B.$$

27. To be proved: $\lim_{x\to1+}\dfrac{1}{x-1}=\infty$. Proof: Let $B>0$ be given. We have $\dfrac{1}{x-1}>B$ if $0<x-1<1/B$, that is, if $1<x<1+\delta$, where $\delta=1/B$. This completes the proof.

28. To be proved: $\lim_{x\to1-}\dfrac{1}{x-1}=-\infty$. Proof: Let $B>0$ be given. We have $\dfrac{1}{x-1}<-B$ if $0>x-1>-1/B$, that is, if $1-\delta<x<1$, where $\delta=1/B$.. This completes the proof.

29. To be proved: $\lim_{x\to\infty}\dfrac{1}{\sqrt{x^2+1}}=0$. Proof: Let $\epsilon>0$ be given. We have

$$\left|\frac{1}{\sqrt{x^2+1}}\right|=\frac{1}{\sqrt{x^2+1}}<\frac{1}{x}<\epsilon$$

provided $x>R$, where $R=1/\epsilon$. This completes the proof.

30. To be proved: $\lim_{x\to\infty}\sqrt{x}=\infty$. Proof: Let $B>0$ be given. We have $\sqrt{x}>B$ if $x>R$ where $R=B^2$. This completes the proof.

31. To be proved: if $\lim_{x\to a}f(x)=L$ and $\lim_{x\to a}f(x)=M$, then $L=M$.
Proof: Suppose $L\neq M$. Let $\epsilon=|L-M|/3$. Then $\epsilon>0$. Since $\lim_{x\to a}f(x)=L$, there exists $\delta_1>0$ such that $|f(x)-L|<\epsilon$ if $|x-a|<\delta_1$. Since $\lim_{x\to a}f(x)=M$, there exists $\delta_2>0$ such that $|f(x)-M|<\epsilon$ if $|x-a|<\delta_2$. Let $\delta=\min(\delta_1,\delta_2)$. If $|x-a|<\delta$, then

$$\begin{aligned}3\epsilon=|L-M|&=|(f(x)-M)+(L-f(x)|\\&\le|f(x)-M|+|f(x)-L|<\epsilon+\epsilon=2\epsilon.\end{aligned}$$

This implies that $3<2$, a contradiction. Thus the original assumption that $L\neq M$ must be incorrect. Therefore $L=M$.

32. To be proved: if $\lim_{x\to a} g(x) = M$, then there exists $\delta > 0$ such that if $0 < |x-a| < \delta$, then $|g(x)| < 1 + |M|$.
Proof: Taking $\epsilon = 1$ in the definition of limit, we obtain a number $\delta > 0$ such that if $0 < |x-a| < \delta$, then $|g(x) - M| < 1$. It follows from this latter inequality that

$$|g(x)| = |(g(x)-M)+M| \le |G(x)-M|+|M| < 1+|M|.$$

33. To be proved: if $\lim_{x\to a} f(x) = L$ and $\lim_{x\to a} g(x) = M$, then $\lim_{x\to a} f(x)g(x) = LM$.
Proof: Let $\epsilon > 0$ be given. Since $\lim_{x\to a} f(x) = L$, there exists $\delta_1 > 0$ such that $|f(x) - L| < \epsilon/(2(1+|M|))$ if $0 < |x-a| < \delta_1$. Since $\lim_{x\to a} g(x) = M$, there exists $\delta_2 > 0$ such that $|g(x) - M| < \epsilon/(2(1+|L|))$ if $0 < |x-a| < \delta_2$. By Exercise 32, there exists $\delta_3 > 0$ such that $|g(x)| < 1 + |M|$ if $0 < |x-a| < \delta_3$. Let $\delta = \min(\delta_1, \delta_2, \delta_3)$. If $|x-a| < \delta$, then

$$\begin{aligned}
|f(x)g(x) - LM &= |f(x)g(x) - Lg(x) + Lg(x) - LM| \\
&= |(f(x)-L)g(x) + L(g(x)-M)| \\
&\le |(f(x)-L)g(x)| + |L(g(x)-M)| \\
&= |f(x)-L||g(x)| + |L||g(x)-M| \\
&< \frac{\epsilon}{2(1+|M|)}(1+|M|) + |L|\frac{\epsilon}{2(1+|L|)} \\
&\le \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.
\end{aligned}$$

Thus $\lim_{x\to a} f(x)g(x) = LM$.

34. To be proved: if $\lim_{x\to a} g(x) = M$ where $M \ne 0$, then there exists $\delta > 0$ such that if $0 < |x-a| < \delta$, then $|g(x)| > |M|/2$.
Proof: By the definition of limit, there exists $\delta > 0$ such that if $0 < |x-a| < \delta$, then $|g(x) - M| < |M|/2$ (since $|M|/2$ is a positive number). This latter inequality implies that

$$|M| = |g(x)+(M-g(x))| \le |g(x)|+|g(x)-M| < |g(x)|+\frac{|M|}{2}.$$

It follows that $|g(x)| > |M| - (|M|/2) = |M|/2$, as required.

35. To be proved: if $\lim_{x\to a} g(x) = M$ where $M \ne 0$, then $\lim_{x\to a} \frac{1}{g(x)} = \frac{1}{M}$.
Proof: Let $\epsilon > 0$ be given. Since $\lim_{x\to a} g(x) = M \ne 0$, there exists $\delta_1 > 0$ such that $|g(x) - M| < \epsilon|M|^2/2$ if $0 < |x-a| < \delta_1$. By Exercise 34, there exists $\delta_2 > 0$ such that $|g(x)| > |M|/2$ if $0 < |x-a| < \delta_3$. Let $\delta = \min(\delta_1, \delta_2)$. If $0 < |x-a| < \delta$, then

$$\left|\frac{1}{g(x)} - \frac{1}{M}\right| = \frac{|M-g(x)|}{|M||g(x)|} < \frac{\epsilon|M|^2}{2}\,\frac{2}{|M|^2} = \epsilon.$$

This completes the proof.

36. To be proved: if $\lim_{x\to a} f(x) = L$ and $\lim_{x\to a} f(x) = M \ne 0$, then $\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{L}{M}$.
Proof: By Exercises 33 and 35 we have

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} f(x) \times \frac{1}{g(x)} = L \times \frac{1}{M} = \frac{L}{M}.$$

37. To be proved: if f is continuous at L and $\lim_{x\to c} g(x) = L$, then $\lim_{x\to c} f(g(x)) = f(L)$.
Proof: Let $\epsilon > 0$ be given. Since f is continuous at L, there exists a number $\gamma > 0$ such that if $|y - L| < \gamma$, then $|f(y) - f(L)| < \epsilon$. Since $\lim_{x\to c} g(x) = L$, there exists $\delta > 0$ such that if $0 < |x-c| < \delta$, then $|g(x) - L| < \gamma$. Taking $y = g(x)$, it follows that if $0 < |x-c| < \delta$, then $|f(g(x)) - f(L)| < \epsilon$, so that $\lim_{x\to c} f(g(x)) = f(L)$.

38. To be proved: if $f(x) \le g(x) \le h(x)$ in an open interval containing $x = a$ (say, for $a - \delta_1 < x < a + \delta_1$, where $\delta_1 > 0$), and if $\lim_{x\to a} f(x) = \lim_{x\to a} h(x) = L$, then also $\lim_{x\to a} g(x) = L$.
Proof: Let $\epsilon > 0$ be given. Since $\lim_{x\to a} f(x) = L$, there exists $\delta_2 > 0$ such that if $0 < |x-a| < \delta_2$, then $|f(x) - L| < \epsilon/3$. Since $\lim_{x\to a} h(x) = L$, there exists $\delta_3 > 0$ such that if $0 < |x-a| < \delta_3$, then $|h(x)-L| < \epsilon/3$. Let $\delta = \min(\delta_1, \delta_2, \delta_3)$. If $0 < |x-a| < \delta$, then

$$\begin{aligned}
|g(x) - L| &= |g(x) - f(x) + f(x) - L| \\
&\le |g(x) - f(x)| + |f(x) - L| \\
&\le |h(x) - f(x)| + |f(x) - L| \\
&= |h(x) - L + L - f(x)| + |f(x) - L| \\
&\le |h(x) - L| + |f(x) - L| + |f(x) - L| \\
&< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon.
\end{aligned}$$

Thus $\lim_{x\to a} g(x) = L$.

Review Exercises 1 (page 93)

1. The average rate of change of x^3 over $[1, 3]$ is

$$\frac{3^3 - 1^3}{3 - 1} = \frac{26}{2} = 13.$$

2. The average rate of change of $1/x$ over $[-2, -1]$ is

$$\frac{(1/(-1)) - (1/(-2))}{-1 - (-2)} = \frac{-1/2}{1} = -\frac{1}{2}.$$

3. The rate of change of x^3 at $x = 2$ is

$$\lim_{h\to 0} \frac{(2+h)^3 - 2^3}{h} = \lim_{h\to 0} \frac{8 + 12h + 6h^2 + h^3 - 8}{h} = \lim_{h\to 0} (12 + 6h + h^2) = 12.$$

4. The rate of change of $1/x$ at $x = -3/2$ is

$$\lim_{h\to 0} \frac{\dfrac{1}{-(3/2)+h} - \left(\dfrac{1}{-3/2}\right)}{h} = \lim_{h\to 0} \frac{\dfrac{2}{2h-3} + \dfrac{2}{3}}{h} = \lim_{h\to 0} \frac{2(3+2h-3)}{3(2h-3)h} = \lim_{h\to 0} \frac{4}{3(2h-3)} = -\frac{4}{9}.$$

5. $\lim_{x\to 1} (x^2 - 4x + 7) = 1 - 4 + 7 = 4$

6. $\lim_{x\to 2} \dfrac{x^2}{1 - x^2} = \dfrac{2^2}{1 - 2^2} = -\dfrac{4}{3}$

7. $\lim_{x\to 1} \dfrac{x^2}{1 - x^2}$ does not exist. The denominator approaches 0 (from both sides) while the numerator does not.

8. $\lim_{x\to 2} \dfrac{x^2 - 4}{x^2 - 5x + 6} = \lim_{x\to 2} \dfrac{(x-2)(x+2)}{(x-2)(x-3)} = \lim_{x\to 2} \dfrac{x+2}{x-3} = -4$

9. $\lim_{x\to 2} \dfrac{x^2 - 4}{x^2 - 4x + 4} = \lim_{x\to 2} \dfrac{(x-2)(x+2)}{(x-2)^2} = \lim_{x\to 2} \dfrac{x+2}{x-2}$ does not exist. The denominator approaches 0 (from both sides) while the numerator does not.

10. $\lim_{x\to 2-} \dfrac{x^2 - 4}{x^2 - 4x + 4} = \lim_{x\to 2-} \dfrac{x+2}{x-2} = -\infty$

11. $\lim_{x\to -2+} \dfrac{x^2 - 4}{x^2 + 4x + 4} = \lim_{x\to -2+} \dfrac{x-2}{x+2} = -\infty$

12. $\lim_{x\to 4} \dfrac{2 - \sqrt{x}}{x - 4} = \lim_{x\to 4} \dfrac{4 - x}{(2 + \sqrt{x})(x - 4)} = -\dfrac{1}{4}$

13. $$\lim_{x\to 3} \frac{x^2 - 9}{\sqrt{x} - \sqrt{3}} = \lim_{x\to 3} \frac{(x-3)(x+3)(\sqrt{x} + \sqrt{3})}{x - 3} = \lim_{x\to 3} (x+3)(\sqrt{x} + \sqrt{3}) = 12\sqrt{3}$$

14. $$\lim_{h\to 0} \frac{h}{\sqrt{x+3h} - \sqrt{x}} = \lim_{h\to 0} \frac{h(\sqrt{x+3h} + \sqrt{x})}{(x+3h) - x} = \lim_{h\to 0} \frac{\sqrt{x+3h} + \sqrt{x}}{3} = \frac{2\sqrt{x}}{3}$$

15. $\lim_{x\to 0+} \sqrt{x - x^2} = 0$

16. $\lim_{x\to 0} \sqrt{x - x^2}$ does not exist because $\sqrt{x - x^2}$ is not defined for $x < 0$.

17. $\lim_{x\to 1} \sqrt{x - x^2}$ does not exist because $\sqrt{x - x^2}$ is not defined for $x > 1$.

18. $\lim_{x\to 1-} \sqrt{x - x^2} = 0$

19. $\lim_{x\to \infty} \dfrac{1 - x^2}{3x^2 - x - 1} = \lim_{x\to \infty} \dfrac{(1/x^2) - 1}{3 - (1/x) - (1/x^2)} = -\dfrac{1}{3}$

20. $\lim_{x\to -\infty} \dfrac{2x + 100}{x^2 + 3} = \lim_{x\to -\infty} \dfrac{(2/x) + (100/x^2)}{1 + (3/x^2)} = 0$

21. $\lim_{x\to -\infty} \dfrac{x^3 - 1}{x^2 + 4} = \lim_{x\to -\infty} \dfrac{x - (1/x^2)}{1 + (4/x^2)} = -\infty$

22. $\lim_{x\to \infty} \dfrac{x^4}{x^2 - 4} = \lim_{x\to \infty} \dfrac{x^2}{1 - (4/x^2)} = \infty$

23. $\lim_{x\to 0+} \dfrac{1}{\sqrt{x - x^2}} = \infty$

24. $\lim_{x\to 1/2} \dfrac{1}{\sqrt{x - x^2}} = \dfrac{1}{\sqrt{1/4}} = 2$

25. $\lim_{x\to \infty} \sin x$ does not exist; $\sin x$ takes the values -1 and 1 in any interval (R, ∞), and limits, if they exist, must be unique.

26. $\lim_{x\to \infty} \dfrac{\cos x}{x} = 0$ by the squeeze theorem, since

$$-\frac{1}{x} \le \frac{\cos x}{x} \le \frac{1}{x} \quad \text{for all } x > 0$$

and $\lim_{x\to\infty}(-1/x) = \lim_{x\to\infty}(1/x) = 0$.

27. $\lim_{x\to 0} x \sin \dfrac{1}{x} = 0$ by the squeeze theorem, since

$$-|x| \le x \sin \frac{1}{x} \le |x| \quad \text{for all } x \ne 0$$

and $\lim_{x\to 0}(-|x|) = \lim_{x\to 0} |x| = 0$.

28. $\lim_{x\to 0} \sin \dfrac{1}{x^2}$ does not exist; $\sin(1/x^2)$ takes the values -1 and 1 in any interval $(-\delta, \delta)$, where $\delta > 0$, and limits, if they exist, must be unique.

29. $$\begin{aligned}\lim_{x\to -\infty} [x + \sqrt{x^2 - 4x + 1}] &= \lim_{x\to -\infty} \frac{x^2 - (x^2 - 4x + 1)}{x - \sqrt{x^2 - 4x + 1}} \\ &= \lim_{x\to -\infty} \frac{4x - 1}{x - |x|\sqrt{1 - (4/x) + (1/x^2)}} \\ &= \lim_{x\to -\infty} \frac{x[4 - (1/x)]}{x + x\sqrt{1 - (4/x) + (1/x^2)}} \\ &= \lim_{x\to -\infty} \frac{4 - (1/x)}{1 + \sqrt{1 - (4/x) + (1/x^2)}} = 2.\end{aligned}$$

Note how we have used $|x| = -x$ (in the second last line), because $x \to -\infty$.

30. $\lim_{x\to \infty} [x + \sqrt{x^2 - 4x + 1}] = \infty + \infty = \infty$

31. $f(x) = x^3 - 4x^2 + 1$ is continuous on the whole real line and so is discontinuous nowhere.

32. $f(x) = \dfrac{x}{x+1}$ is continuous everywhere on its domain, which consists of all real numbers except $x = -1$. It is discontinuous nowhere.

33. $f(x) = \begin{cases} x^2 & \text{if } x > 2 \\ x & \text{if } x \le 2 \end{cases}$ is defined everywhere and discontinuous at $x = 2$, where it is, however, left continuous since $\lim_{x\to 2-} f(x) = 2 = f(2)$.

34. $f(x) = \begin{cases} x^2 & \text{if } x > 1 \\ x & \text{if } x \le 1 \end{cases}$ is defined and continuous everywhere, and so discontinuous nowhere. Observe that $\lim_{x\to 1-} f(x) = 1 = \lim_{x\to 1+} f(x)$.

35. $f(x) = H(x-1) = \begin{cases} 1 & \text{if } x \ge 1 \\ 0 & \text{if } x < 1 \end{cases}$ is defined everywhere and discontinuous at $x = 1$ where it is, however, right continuous.

36. $f(x) = H(9-x^2) = \begin{cases} 1 & \text{if } -3 \le x \le 3 \\ 0 & \text{if } x < -3 \text{ or } x > 3 \end{cases}$ is defined everywhere and discontinuous at $x = \pm 3$. It is right continuous at -3 and left continuous at 3.

37. $f(x) = |x| + |x+1|$ is defined and continuous everywhere. It is discontinuous nowhere.

38. $f(x) = \begin{cases} |x|/|x+1| & \text{if } x \ne -1 \\ 1 & \text{if } x = -1 \end{cases}$ is defined everywhere and discontinuous at $x = -1$ where it is neither left nor right continuous since $\lim_{x\to -1} f(x) = \infty$, while $f(-1) = 1$.

Challenging Problems 1 (page 94)

1. Let $0 < a < b$. The average rate of change of x^3 over $[a, b]$ is

$$\frac{b^3 - a^3}{b - a} = b^2 + ab + a^2.$$

The instantaneous rate of change of x^3 at $x = c$ is

$$\lim_{h\to 0} \frac{(c+h)^3 - c^3}{h} = \lim_{h\to 0} \frac{3c^2h + 3ch^2 + h^3}{h} = 3c^2.$$

If $c = \sqrt{(a^2 + ab + b^2)/3}$, then $3c^2 = a^2 + ab + b^2$, so the average rate of change over $[a, b]$ is the instantaneous rate of change at $\sqrt{(a^2 + ab + b^2)/3}$.

Claim: $\sqrt{(a^2 + ab + b^2)/3} > (a+b)/2$.

Proof: Since $a^2 - 2ab + b^2 = (a-b)^2 > 0$, we have

$$\begin{aligned} 4a^2 + 4ab + 4b^2 &> 3a^2 + 6ab + 3b^2 \\ \frac{a^2 + ab + b^2}{3} &> \frac{a^2 + 2ab + b^2}{4} = \left(\frac{a+b}{2}\right)^2 \\ \sqrt{\frac{a^2 + ab + b^2}{3}} &> \frac{a+b}{2}. \end{aligned}$$

2. For x near 0 we have $|x-1| = 1 - x$ and $|x+1| = x + 1$. Thus

$$\lim_{x\to 0} \frac{x}{|x-1| - |x+1|} = \lim_{x\to 0} \frac{x}{(1-x) - (x+1)} = -\frac{1}{2}.$$

3. For x near 3 we have $|5 - 2x| = 2x - 5$, $|x - 2| = x - 2$, $|x - 5| = 5 - x$, and $|3x - 7| = 3x - 7$. Thus

$$\begin{aligned} \lim_{x\to 3} \frac{|5-2x| - |x-2|}{|x-5| - |3x-7|} &= \lim_{x\to 3} \frac{2x - 5 - (x-2)}{5 - x - (3x - 7)} \\ &= \lim_{x\to 3} \frac{x-3}{4(3-x)} = -\frac{1}{4}. \end{aligned}$$

4. Let $y = x^{1/6}$. Then we have

$$\begin{aligned} \lim_{x\to 64} \frac{x^{1/3} - 4}{x^{1/2} - 8} &= \lim_{y\to 2} \frac{y^2 - 4}{y^3 - 8} \\ &= \lim_{y\to 2} \frac{(y-2)(y+2)}{(y-2)(y^2 + 2y + 4)} \\ &= \lim_{y\to 2} \frac{y+2}{y^2 + 2y + 4} = \frac{4}{12} = \frac{1}{3}. \end{aligned}$$

5. Use $a - b = \dfrac{a^3 - b^3}{a^2 + ab + b^2}$ to handle the denominator. We have

$$\begin{aligned} &\lim_{x\to 1} \frac{\sqrt{3+x} - 2}{\sqrt[3]{7+x} - 2} \\ &= \lim_{x\to 1} \frac{3 + x - 4}{\sqrt{3+x} + 2} \times \frac{(7+x)^{2/3} + 2(7+x)^{1/3} + 4}{(7+x) - 8} \\ &= \lim_{x\to 1} \frac{(7+x)^{2/3} + 2(7+x)^{1/3} + 4}{\sqrt{3+x} + 2} = \frac{4+4+4}{2+2} = 3. \end{aligned}$$

6. $r_+(a) = \dfrac{-1 + \sqrt{1+a}}{a}$, $r_-(a) = \dfrac{-1 - \sqrt{1+a}}{a}$.

a) $\lim_{a\to 0} r_-(a)$ does not exist. Observe that the right limit is $-\infty$ and the left limit is ∞.

b) From the following table it appears that $\lim_{a\to 0} r_+(a) = 1/2$, the solution of the linear equation $2x - 1 = 0$ which results from setting $a = 0$ in the quadratic equation $ax^2 + 2x - 1 = 0$.

a	$r_+(a)$
1	0.41421
0.1	0.48810
−0.1	0.51317
0.01	0.49876
−0.01	0.50126
0.001	0.49988
−0.001	0.50013

c) $$\lim_{a\to 0} r_+(a) = \lim_{a\to 0} \frac{\sqrt{1+a}-1}{a} = \lim_{a\to 0} \frac{(1+a)-1}{a(\sqrt{1+a}+1)} = \lim_{a\to 0} \frac{1}{\sqrt{1+a}+1} = \frac{1}{2}.$$

7. TRUE or FALSE

a) If $\lim_{x\to a} f(x)$ exists and $\lim_{x\to a} g(x)$ does not exist, then $\lim_{x\to a}\big(f(x)+g(x)\big)$ does not exist.

TRUE, because if $\lim_{x\to a}\big(f(x)+g(x)\big)$ were to exist then

$$\lim_{x\to a} g(x) = \lim_{x\to a}\Big(f(x)+g(x)-f(x)\Big) = \lim_{x\to a}\Big(f(x)+g(x)\Big) - \lim_{x\to a} f(x)$$

would also exist.

b) If neither $\lim_{x\to a} f(x)$ nor $\lim_{x\to a} g(x)$ exists, then $\lim_{x\to a}\big(f(x)+g(x)\big)$ does not exist.

FALSE. Neither $\lim_{x\to 0} 1/x$ nor $\lim_{x\to 0}(-1/x)$ exist, but $\lim_{x\to 0}\big((1/x)+(-1/x)\big) = \lim_{x\to 0} 0 = 0$ exists.

c) If f is continuous at a, then so is $|f|$.

TRUE. For any two real numbers u and v we have

$$\Big||u|-|v|\Big| \le |u-v|.$$

This follows from

$$|u| = |u-v+v| \le |u-v|+|v|, \quad \text{and}$$
$$|v| = |v-u+u| \le |v-u|+|u| = |u-v|+|u|.$$

Now we have

$$\Big||f(x)|-|f(a)|\Big| \le |f(x)-f(a)|$$

so the left side approaches zero whenever the right side does. This happens when $x\to a$ by the continuity of f at a.

d) If $|f|$ is continuous at a, then so is f.

FALSE. The function $f(x) = \begin{cases} -1 & \text{if } x<0 \\ 1 & \text{if } x\ge 0\end{cases}$ is discontinuous at $x=0$, but $|f(x)|=1$ everywhere, and so is continuous at $x=0$.

e) If $f(x) < g(x)$ in an interval around a and if $\lim_{x\to a} f(x) = L$ and $\lim_{x\to a} g(x) = M$ both exist, then $L < M$.

FALSE. Let $g(x) = \begin{cases} x^2 & \text{if } x\ne 0 \\ 1 & \text{if } x=0\end{cases}$ and let $f(x) = -g(x)$. Then $f(x) < g(x)$ for all x, but $\lim_{x\to 0} f(x) = 0 = \lim_{x\to 0} g(x)$. (Note: under the given conditions, it is TRUE that $L \le M$, but not necessarily true that $L < M$.)

8. a) To be proved: if f is a continuous function defined on a closed interval $[a,b]$, then the range of f is a closed interval.

Proof: By the Max-Min Theorem there exist numbers u and v in $[a,b]$ such that $f(u)\le f(x)\le f(v)$ for all x in $[a,b]$. By the Intermediate-Value Theorem, $f(x)$ takes on all values between $f(u)$ and $f(v)$ at values of x between u and v, and hence at points of $[a,b]$. Thus the range of f is $[f(u), f(v)]$, a closed interval.

b) If the domain of the continuous function f is an open interval, the range of f can be any interval (open, closed, half open, finite, or infinite).

9. $$f(x) = \frac{x^2-1}{|x^2-1|} = \begin{cases} -1 & \text{if } -1<x<1 \\ 1 & \text{if } x<-1 \text{ or } x>1\end{cases}.$$
f is continuous wherever it is defined, that is at all points except $x=\pm 1$. f has left and right limits -1 and 1, respectively, at $x=1$, and has left and right limits 1 and -1, respectively, at $x=-1$. It is not, however, discontinuous at any point, since -1 and 1 are not in its domain.

10. $$f(x) = \frac{1}{x-x^2} = \frac{1}{\frac14 - \left(\frac14 - x + x^2\right)} = \frac{1}{\frac14 - \left(x-\frac12\right)^2}.$$
Observe that $f(x) \ge f(1/2) = 4$ for all x in $(0,1)$.

11. Suppose f is continuous on $[0,1]$ and $f(0) = f(1)$.

a) To be proved: $f(a) = f(a+\frac12)$ for some a in $[0,\frac12]$.

Proof: If $f(1/2) = f(0)$ we can take $a=0$ and be done. If not, let

$$g(x) = f(x+\tfrac12) - f(x).$$

Then $g(0)\ne 0$ and

$$g(1/2) = f(1)-f(1/2) = f(0)-f(1/2) = -g(0).$$

Since g is continuous and has opposite signs at $x=0$ and $x=1/2$, the Intermediate-Value Theorem assures us that there exists a between 0 and 1/2 such that $g(a)=0$, that is, $f(a) = f(a+\frac12)$.

b) To be proved: if $n>2$ is an integer, then $f(a) = f(a+\frac1n)$ for some a in $[0,1-\frac1n]$.

Proof: Let $g(x) = f(x+\frac1n)-f(x)$. Consider the numbers $x=0$, $x=1/n$, $x=2/n$, $\ldots$, $x=(n-1)/n$. If $g(x)=0$ for any of these numbers, then we can let a be that number. Otherwise, $g(x)\ne 0$ at any of these numbers. Suppose that the values of g at all these numbers has the same sign (say positive). Then we have

$$f(1) > f(\tfrac{n-1}{n}) > \cdots > f(\tfrac2n) > \tfrac1n > f(0),$$

which is a contradiction, since $f(0)=f(1)$. Therefore there exists j in the set $\{0,1,2,\ldots,n-1\}$ such that $g(j/n)$ and $g((j+1)/n)$ have opposite sign. By the Intermediate-Value Theorem, $g(a)=0$ for some a between j/n and $(j+1)/n$, which we had to prove.