Functions and Models 1

1.1 Four Ways to Represent a Function

Suggested Time and Emphasis

1 class Essential material

Points to Stress

1. The definition of function as a rule, and the four representations (visual, algebraic, numeric, descriptive).

2. The definitions of terms associated with functions, such as domain, range, increasing, and decreasing.

3. Piecewise defined functions.

4. Symmetry: even and odd functions.

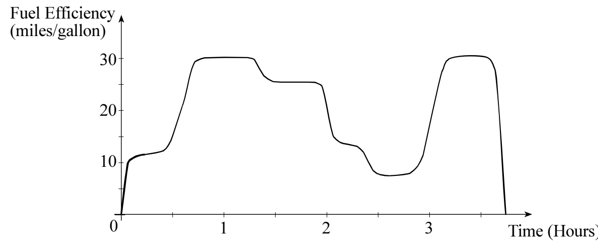
Text Discussion

- Why does the author assert that "the $\sqrt{x}$ key on your calculator is not quite the same as the exact mathematical function f defined by $f(x) = \sqrt{x}$"?
- How do the intuitive definitions of increasing and decreasing translate into the mathematical definitions of these terms presented on page 21?

Materials for Lecture

- Show how the step function described in the text does pass the Vertical Line Test.
- Draw the distinction between exact and approximate function values, as done in Example 1.
- Present graphs of even and odd functions, such as $\sin x$, $\cos x + x^2$, and $\cos(\sin x)$, and check with the standard algebraic tests.
- Draw a graph of electrical power consumption in the classroom versus time on a typical weekday, pointing out important features throughout, and using the vocabulary of this section as much as possible.

- Draw a graph of fuel efficiency versus time on a trip, such as the one below. Lead a discussion of what could have happened on the trip.



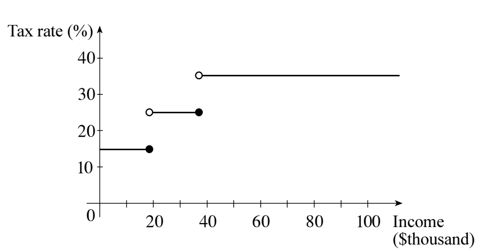
ANSWER Fuel efficiency is high when the car is traveling at a steady and moderate speed, lower when the car is accelerating or moving at more than 40 mi/h.

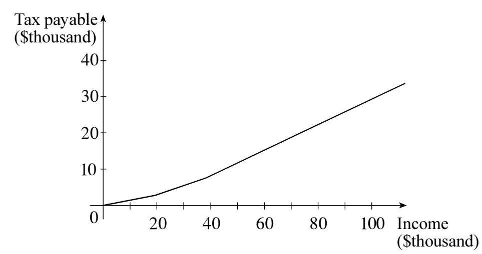
- Discuss the domain and range of $f(x) = \dfrac{1}{\sqrt{x^2 - 1}}$.

Workshop/Discussion

- In 1984, United States President Ronald Reagan proposed a plan to change the United States personal income tax system. According to his plan, the income tax would be 15% on the first \$19,300 earned, 25% on the next \$18,800, and 35% on all income above and beyond that. Describe this situation to the class, and have them graph (marginal) tax rate and tax owed versus income for incomes ranging from \$0 to \$80,000. Then have them try to come up with equations describing this situation.

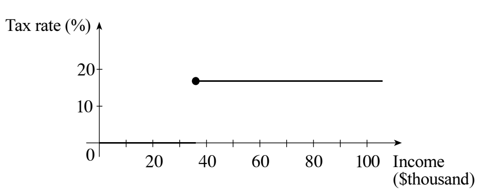
ANSWER

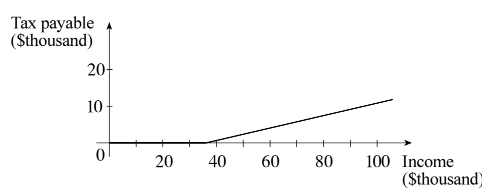




- In the year 2000, Presidential candidate Steve Forbes proposed a "flat tax" model: 0% on the first \$36,000 and 17% on the rest. Have the students do the same analysis, and compare the two models. As an extension, perhaps have the students look at a current tax table and draw similar graphs.

ANSWER





- Discuss the domain and range of $f(x) = \dfrac{x-2}{x^2 - x - 12}$ and $g(x) = 1 - |x|$. Discuss where the functions are increasing and decreasing, and if they are even or odd.

- Discuss the domain and range of a function such as $f(x) = \begin{cases} \sqrt{x} & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational} \end{cases}$

  Also talk about why f is neither increasing nor decreasing for $x > 0$. Stress that when dealing with new sorts of functions, it becomes important to know the precise mathematical definitions of such terms.

Group Work 1: Every Picture Tells a Story

Put the students in groups of four, and have them work on the exercise. If there are questions, encourage them to ask each other before asking you. After going through the correct matching with them, have each group tell their story to the class and see if it fits the remaining graph.

ANSWERS **1.** (b) **2.** (a) **3.** (c) **4.** The roast was cooked in the morning and put in the refrigerator in the afternoon.

Group Work 2: Finding a Formula

Make sure that the students know the equation of a circle with radius r, and that they remember the notation for piecewise-defined functions. Split the students into groups of four. In each group, have half of the students work on each problem first, and then have them check each other's work. If the students find these problems difficult, have them work together on each problem.

ANSWERS **1.** $f(x) = \begin{cases} -x-2 & \text{if } x \le -2 \\ x+2 & \text{if } -2 < x \le 0 \\ 2 & \text{if } x > 0 \end{cases}$ **2.** $g(x) = \begin{cases} x+4 & \text{if } x \le -2 \\ 2 & \text{if } -2 < x \le 0 \\ \sqrt{4-x^2} & \text{if } 0 < x \le 2 \\ x-2 & \text{if } x > 2 \end{cases}$

Homework Problems

CORE EXERCISES 1, 2, 4, 6, 7, 9, 18, 22, 28, 32, 43, 56, 68

SAMPLE ASSIGNMENT 1, 2, 4, 6, 7, 9, 11, 18, 20, 22, 23, 25, 28, 29, 32, 33, 39, 43, 43, 47, 51, 56, 57, 63, 68, 68

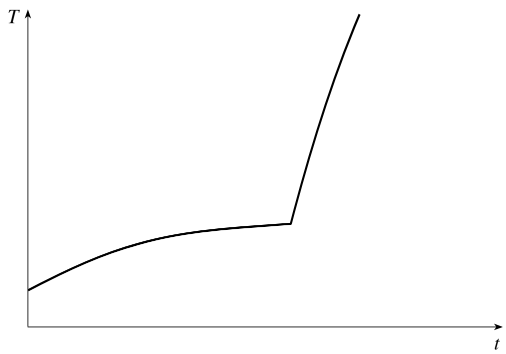
NOTE The workshop example on income tax should be covered if Exercise 61 is assigned.

EXERCISE	D	A	N	G
1				×
2				×
4	×			
6				×
7				×
9				×
11				×
18	×			×
20				×
22		×		×
23		×	×	
25		×	×	
28			×	

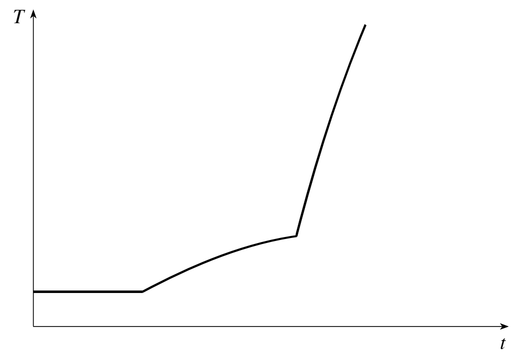
EXERCISE	D	A	N	G
29		×	×	
32		×		×
33		×		
39		×	×	×
43		×		×
43				×
47		×		
51		×		×
56		×		
57		×		
63				×
68		×		×
68		×		×

GROUP WORK 1, SECTION 1.1
Every Picture Tells a Story

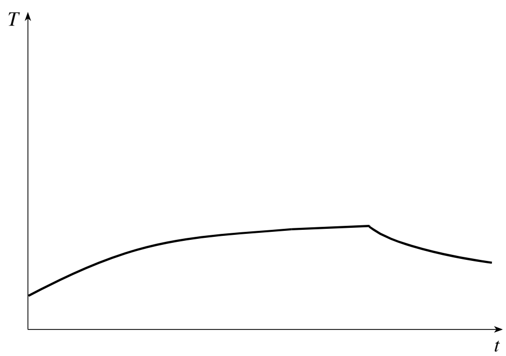
One of the skills you will be learning in this course is the ability to take a description of a real-world occurrence, and translate it into mathematics. Conversely, given a mathematical description of a phenomenon, you will learn how to describe what is happening in plain language. Here follow four graphs of temperature versus time and three stories. Match the stories with the graphs. When finished, write a similar story that would correspond to the final graph.



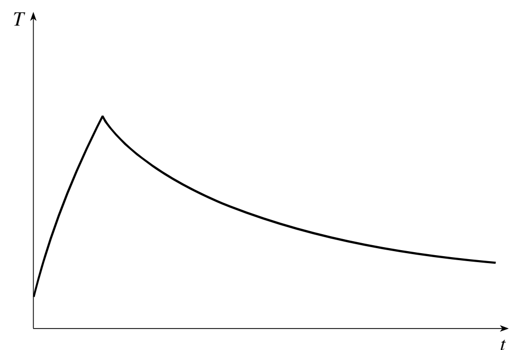
GRAPH 1



GRAPH 2



GRAPH 3



GRAPH 4

(a) I took my roast out of the freezer at noon, and left it on the counter to thaw. Then I cooked it in the oven when I got home.

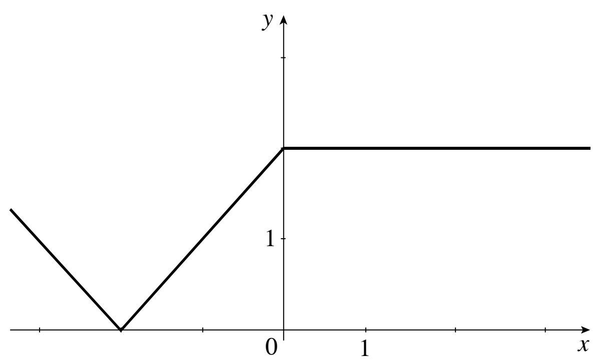
(b) I took my roast out of the freezer this morning, and left it on the counter to thaw. Then I cooked it in the oven when I got home.

(c) I took my roast out of the freezer this morning, and left it on the counter to thaw. I forgot about it, and went out for Chinese food on my way home from work. I put the roast in the refrigerator when I finally got home.

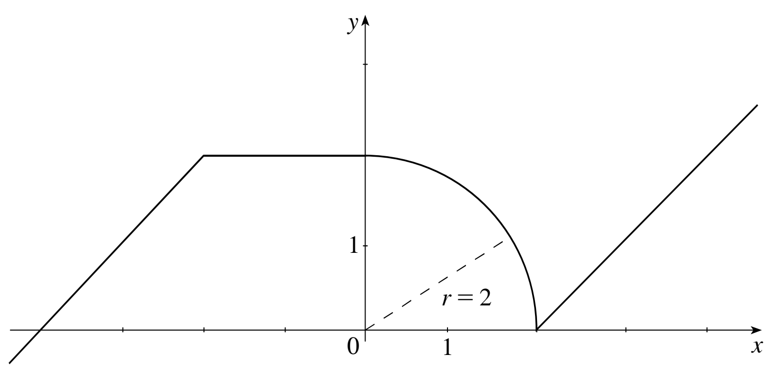
GROUP WORK 2, SECTION 1.1
Finding a Formula

Find formulas for the following functions:

1.



2.



1.2 Mathematical Models: A Catalog of Essential Functions

Additional material relevant to this section is contained in Appendices B and C.

Suggested Time and Emphasis

1 class   Recommended material

Points to Stress

1. The modeling process: developing, analyzing, and interpreting a mathematical model.

2. Interpolation and extrapolation, including the dangers of the latter.

3. Classes of functions: linear, power, rational, algebraic, trigonometric, exponential and transcendental functions. Include the special characteristics of each class of functions.

Text Discussion

- What is the difference between interpolation and extrapolation?

- What is the difference between a power function x^n with $n = 3$ and a cubic function?
  ANSWER A cubic function can have lower order terms, whereas a power function has just one term.

Materials for Lecture

- Point out that linear functions have constant differences in y-values for equally spaced x-values, and use this idea to show that the data set in Table 1 is best modeled by a linear function.

- Review the basic idea of linear regression, perhaps using the log table below to show the use of interpolation and the possible problems with extrapolation. Use the difference between $y(2.0)$ and $y(2.2)$ to estimate $y(2.1)$ (a good approximation) and $y(1)$ (a poor approximation of $\ln(1) = 0$).

x	y
2.0	0.6931
2.2	0.7885
2.4	0.8755
2.6	0.9555
2.8	1.0296
3.0	1.0986

- Work out a model with a quadratic fit, such as the one below. Discuss why a linear model is a poor choice.

x	y
0	-3
1	-2
2	-1
3	-1.5
4	-5

- Discuss the shape, symmetries, and general "flatness" near 0 of the power functions x^n for various values of n. Similarly discuss $\sqrt[n]{x}$ for n even and n odd. A blackline master is provided at the end of this section, before the group work handouts.

Workshop/Discussion

- Most graphing calculators can now do regression analysis. It may be worth class time to have students explore using their calculators to do linear, quadratic, and exponential regressions.
- Exercises 15–18 are all exercises that would be good to go over in class, getting suggestions for models from the students.
- Have the students graph 2^x, $\sin x$, $\sin 2^x$, and $2^{\sin x}$. Discuss why the latter two look the way that they do.
- Discuss the relationship among the graphs of $f(x) = 2^x$, $g(x) = (0.5)^x$, $h(x) = 2^{-x}$, and $k(x) = 4^x$.
- Help the students get a feel for functions and linear growth by having them guess if a given relationship is linear. For example: A person's height versus time? A person's age in seconds versus his or her age in minutes? Gas mileage versus speed? Average temperature versus degrees latitude? Population of Canada versus time? Momentum of a car versus its speed?
  ANSWER Age in seconds versus age in minutes and momentum versus speed are linear relationships.

Group Work 1: Rounding the Bases

On the board, review how to compute the percentage error when estimating π by $\frac{22}{7}$. (Answer: 0.04%) Have them work on the problem in groups. If a group finishes early, have them look at $h(7)$ and $h(10)$ to see how fast the error grows.

ANSWERS 1. $17.811434627, 17, 4.56\%$ **2.** $220.08649875, 201, 8.67\%$ **3.** $45.4314240633, 32, 29.56\%$

Group Work 2: The Small Shall Grow Large

If a group finishes early, ask them to similarly compare x^3 and x^4.

ANSWERS 1. $x^6 \geq x^8$ for $-1 \leq x \leq 1$ **2.** $x^3 \geq x^5$ for $-\infty < x \leq -1, 0 \leq x \leq 1$ **3.** $x^3 \geq x^{105}$ for $-\infty < x \leq -1, 0 \leq x \leq 1$. If the exponents are both even, the answer is the same as for Problem 1, if the exponents are both odd, the answer is the same as for Problem 2.

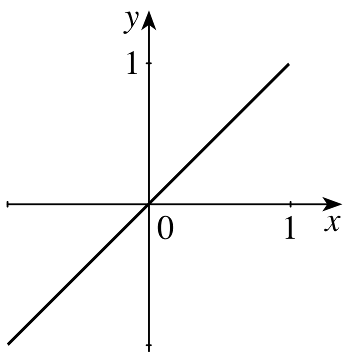
Homework Problems

CORE EXERCISES 1, 4, 10, 13, 15, 18, 22

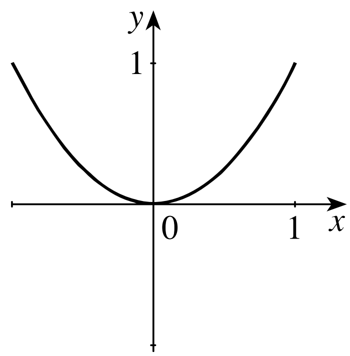
SAMPLE ASSIGNMENT 1, 2, 4, 6, 10, 12, 13, 15, 18, 19, 21, 22, 24

If you assign homework based on regression analysis (Exercises 21–26), the use of the graphing calculator should be discussed in class. (See the attached guide.)

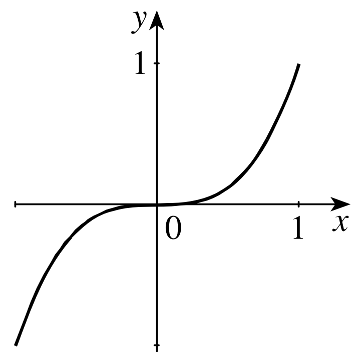
EXERCISE	D	A	N	G
1		×		
2		×		
4				×
6	×	×		×
10	×	×		
12				×
13				×
15	×	×	×	×
18	×	×	×	×
19	×			×
21		×	×	×
22		×		×
24		×		



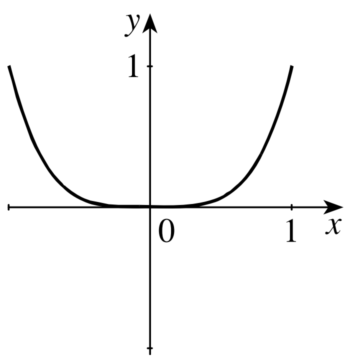
x



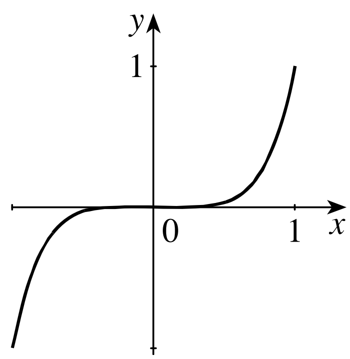
x^2



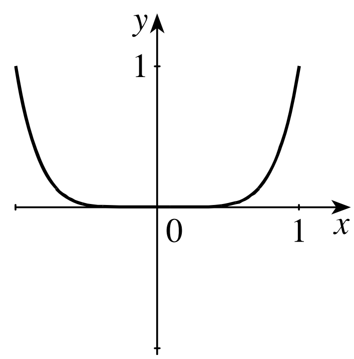
x^3



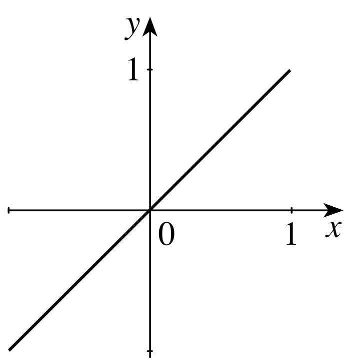
x^4



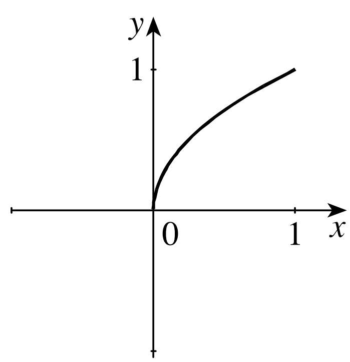
x^5



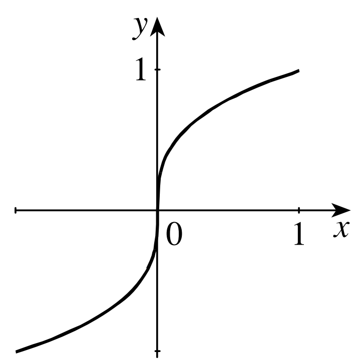
x^6



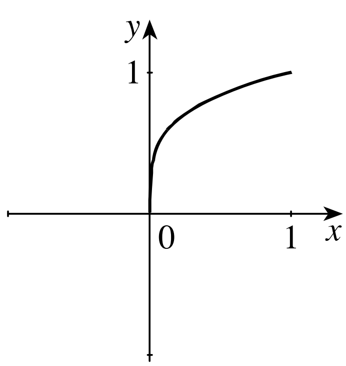
x



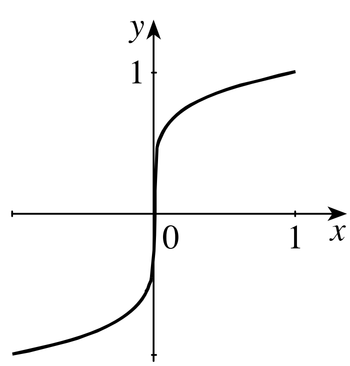
$\sqrt{x}$



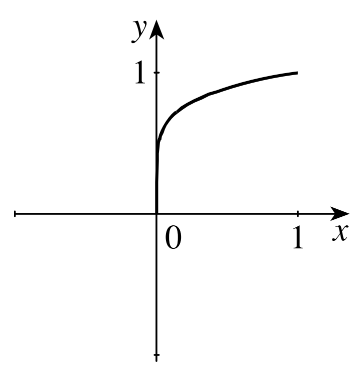
$\sqrt[3]{x}$



$\sqrt[4]{x}$



$\sqrt[5]{x}$



$\sqrt[6]{x}$

GROUP WORK 1, SECTION 1.2
Rounding the Bases

1. For computational efficiency and speed, we often round off constants in equations. For example, consider the linear function

$$f(x) = 3.137619523x + 2.123337012$$

In theory, it is very easy and quick to find $f(1)$, $f(2)$, $f(3)$, $f(4)$, and $f(5)$. In practice, most people doing this computation would probably substitute

$$f(x) = 3x + 2$$

unless a very accurate answer is called for. For example, compute $f(5)$ both ways to see the difference.

The actual value of $f(5)$: __________

The "rounding" estimate: __________

The percentage error: __________

2. Now consider

$$g(x) = 1.12755319x^3 + 3.125694x^2 + 1$$

Again, one is tempted to substitute $g(x) = x^3 + 3x^2 + 1$.

The actual value of $g(5)$: __________

The "rounding" estimate: __________

The percentage error: __________

3. It turns out to be very dangerous to similarly round off exponential functions, due to the nature of their growth. For example, let's look at the function

$$h(x) = (2.145217198123)^x$$

One may be tempted to substitute $h(x) = 2^x$ for this one. Once again, look at the difference between these two functions.

The actual value of $h(5)$: __________

The "rounding" estimate: __________

The percentage error: __________

GROUP WORK 2, SECTION 1.2
The Small Shall Grow Large

1. For what values of x is $x^6 \geq x^8$? For what values is $x^6 \leq x^8$?

2. For what values of x is $x^3 \geq x^5$?

3. For what values of x is $x^3 \geq x^{105}$? Can you generalize your results?

1.3 New Functions from Old Functions

Suggested Time and Emphasis

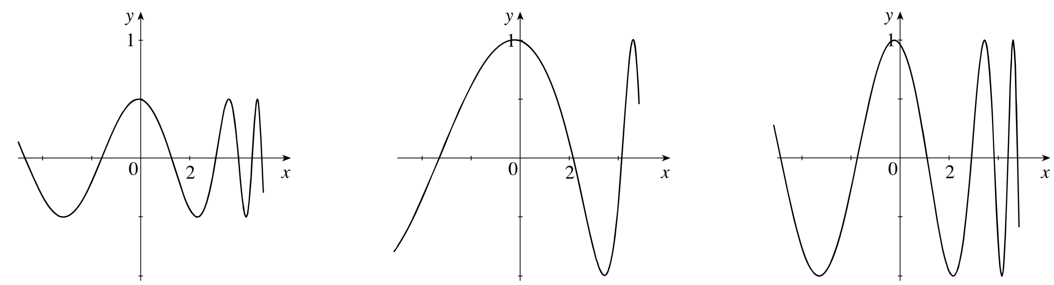
1 class   Essential material

Points to Stress

1. The mechanics and geometry of transforming functions.
2. The mechanics and geometry of adding, subtracting, multiplying, and dividing functions.
3. The mechanics and geometry of composing functions.

Text Discussion

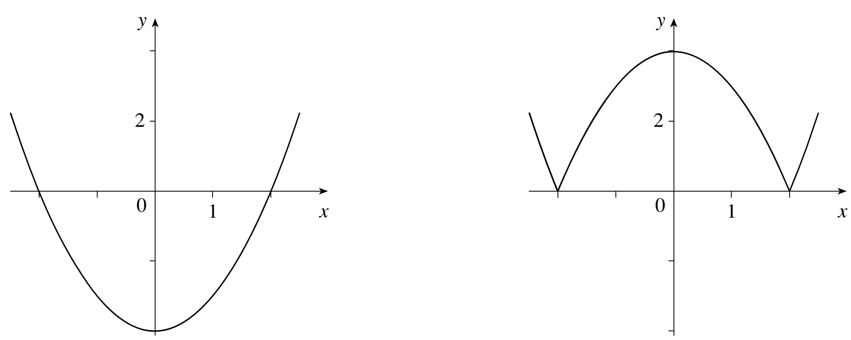
- Label the following graphs: $f(x)$, $\frac{1}{2}f(x)$, $f\left(\frac{1}{2}x\right)$.



ANSWER $\frac{1}{2}f(x)$, $f\left(\frac{1}{2}x\right)$, $f(x)$

- How can we construct the graph of $y = |f(x)|$ from the graph of $y = f(x)$? Explain in words, and demonstrate with the graph of $y = x^2 - 4$.

ANSWER We leave the positive values of $f(x)$ alone, and reflect the negative values about the x-axis.



$y = x^2 - 4$ $y = \left|x^2 - 4\right|$

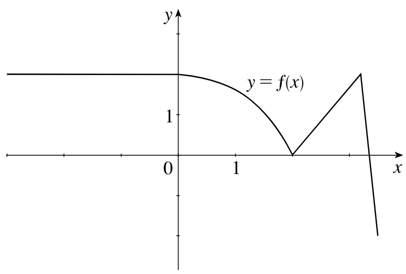
Materials for Lecture

- Present examples of transformation of functions, as in Figure 3 or Example 1, and composition of functions, as in Example 7, and perhaps Exercise 31.
- Using $f(x) = 1/x^2$ and $g(x) = \cos x$, compute the domains of $f + g$, f/g, g/f, $f \circ g$, and $g \circ f$, and the range of $g \circ f$. Pay particular attention to the domain of g/f, as many students will think it is $\mathbb{R}$.

ANSWER $f + g$ has domain $\{x \mid x \neq 0\}$, f/g has domain $\left\{x \mid x \neq 0, x \neq \frac{\pi}{2} + k\pi\right\}$, g/f has domain $\{x \mid x \neq 0\}$, $f \circ g$ has domain $\left\{x \mid x \neq 0, x \neq \frac{\pi}{2} + k\pi\right\}$, $g \circ f$ has domain $\{x \mid x \neq 0\}$, and $g \circ f$ has range $[-1, 1]$.

- Graph $f(x) = \sin(e^x)$ and $g(x) = e^{\sin x}$. Draw the relevant "arrow diagrams" and then write them in the forms $l \circ k$ and $k \circ l$. Then discuss reasons for the differences in their graphs.
  ANSWER See Figure 14 for sample arrow diagrams. f is the a sine function whose argument grows larger faster and faster as we move away from the origin. g is an exponential function whose argument oscillates, causing g to oscillate as well.

- Do the following problem with the students:
  From the graph of $f(x)$ shown below, sketch graphs of $f(2x)$, $2f(x)$, $f(x+2)$, and $f(x-3)$.



- After doing a few basic examples of composition, it is possible to foreshadow the idea of inverses, which will be covered in Section 1.6. Let $f(x) = 2x^3 + 3$ and $g(x) = x^2 - x$. Compute $f \circ g$ and $g \circ f$ for your students. Then ask them to come up with a function $h(x)$ with the property that $(f \circ h)(x) = x$. They may not be used to the idea of coming up with examples for themselves, and so the main hints they will need might be "don't give up," "when in doubt, just try something and see what happens," and "I'm not expecting you to get it in fifteen seconds." If the class is really stuck, have them try $f(x) = 2x^3$ to get a feel for how the game is played. Once they have determined that $h(x) = \sqrt[3]{\frac{x-3}{2}}$, have them compute $(h \circ f)(x)$ and have them conjecture whether, in general, if $(f \circ g)(x) = x$ then $(g \circ f)(x)$ must also equal x.

Workshop/Discussion

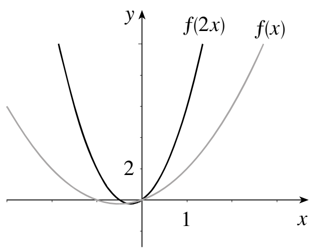
- Have the students graph 3^x, 3^{x+1}, $3^{x+1} - 2$, and $2\left(3^{x+1} - 2\right)$ on their calculators using the same set of axes, first guessing what each graph will look like.

- Have the students use a graphing calculator or computer to graph the following functions, where $f(x) = x + x^2$, first guessing what each graph will look like.

1. $f(2x)$	**4.** $-2f(x)$	**7.** $f(x) + 2$
2. $f\left(\frac{1}{2}x\right)$	**5.** $f(x-2)$	**8.** $2f(2x)$
3. $2f(x)$	**6.** $f(x) - 2$	**9.** $2f\left(\frac{1}{2}x\right)$

ANSWERS

1.



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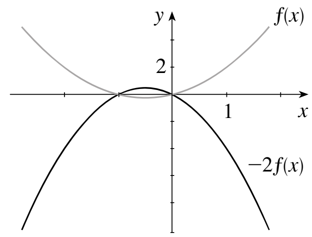

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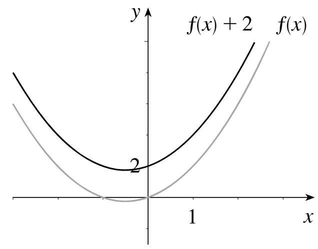

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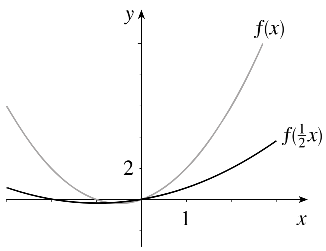

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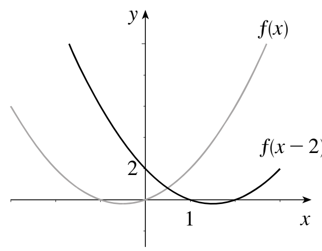

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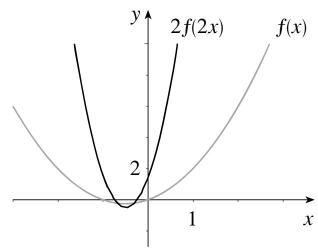
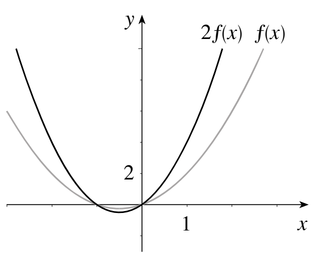


3.



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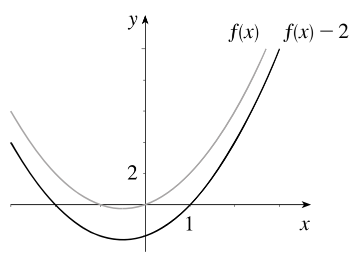
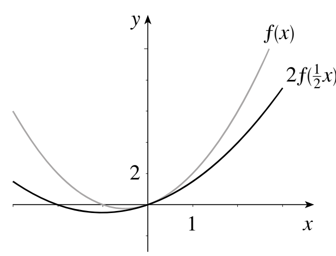


9.

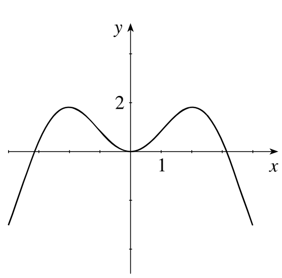


- If $f(x) = x^2$, $g(x) = x+1$, $h(x) = x-1$, and $k(x) = x^3$, ask students to express the following functions as compositions of f, g, h, and k: $y = x^2 + 1$, $y = (x+1)^2$, $y = \left(x^2 - 1\right)^2$, $y = \left[(x+1)^2 + 1\right]^3$, $y = \left[(x-1)^3 + 1\right]^3$.

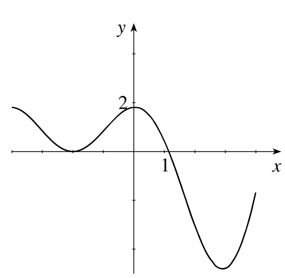
- Discuss the domain and range of f/g and $f \circ g$ if $f(x) = \sin\left(\frac{\pi}{2}x\right)$ and $g(x) = x^2 - 1$.

- Using $f(x) = x \sin x$, explore graphs of $f(x+2)$, $f(x) + 2$, $-f(x)$, $f(-x)$, $|f(x)|$. Note why $f(-x) = f(x)$.

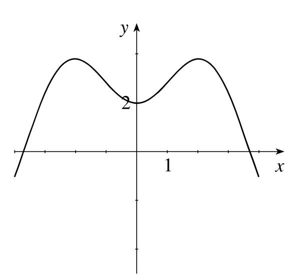
ANSWERS



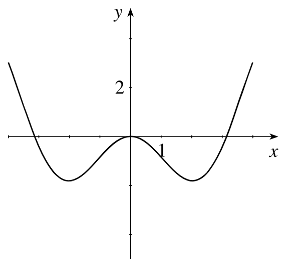
$y = f(x)$



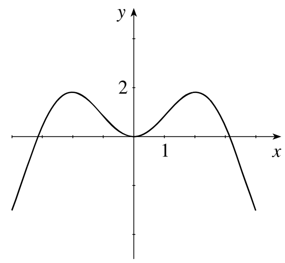
$y = f(x+2)$



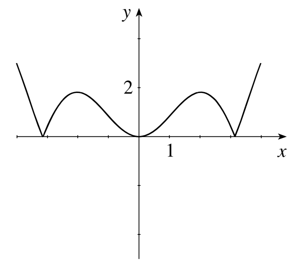
$y = f(x) + 2$



$y = -f(x)$



$y = f(-x)$



$y = |f(x)|$

$f(-x) = f(x)$ because f is even.

Group Work 1: Which is the Original?

ANSWERS **1.** $2f(x+2), 2f(x), f(2x), f(x+2), f(x)$ **2.** $2f(x), f(x), f(x+2), f(2x), 2f(x+2)$

Group Work 2: Label Label Label, I Made It Out of Clay

Some of these transformations are not covered directly in the book. If the students are urged not to give up, and to use the process of elimination and testing individual points, they should be able to complete this activity.

ANSWERS **1.** (d) **2.** (a) **3.** (f) **4.** (e) **5.** (i) **6.** (j) **7.** (b) **8.** (c) **9.** (g) **10.** (h)

Group Work 3: It's More Fun to Compute

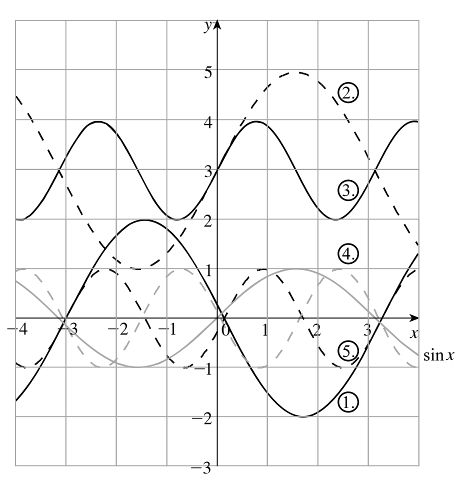
Each group gets one copy of the graph. During each round, one representative from each group stands, and one of the questions below is asked. The representatives write their answer down, and all display their answers at the same time. Each representative has the choice of consulting with their group or not. A correct solo answer is worth two points, and a correct answer after a consult is worth one point.

ANSWERS **1.** 0 **2.** 0 **3.** 1 **4.** 5 **5.** 1 **6.** 1 **7.** 1 **8.** 0 **9.** 2 **10.** 1 **11.** 1 **12.** 1

Group Work 4: Composing Transformations

This exercise is similar to the material in the second Workshop/Discussion example. Students can use TEC Module 1.3 or similar graphing technology to check their answers.

ANSWERS



6. (e) only

Homework Problems

CORE EXERCISES 1, 2, 3, 4, 8, 19, 29, 41, 50, 51, 53, 59

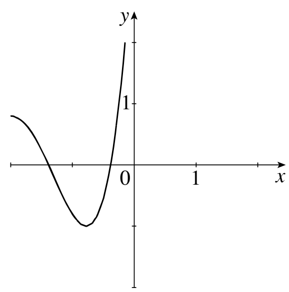
SAMPLE ASSIGNMENT 1, 2, 3, 4, 8, 19, 29, 31, 39, 41, 44, 50, 51, 53, 55, 59, 64

EXERCISE	D	A	N	G
1		×		×
2	×			
3				×
4				×
8	×			×
19				×
29		×		
31		×		
39		×		
41		×		
44		×		
50			×	
51	×			×
53		×		
55	×	×	×	
59	×	×	×	
64	×			

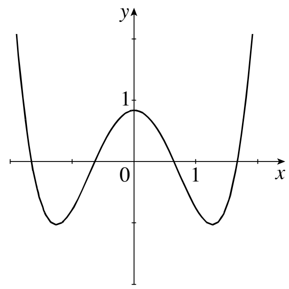
GROUP WORK 1, SECTION 1.3
Which is the Original?

Below are five graphs. One is the graph of a function $f(x)$ and the others include the graphs of $2f(x)$, $f(2x)$, $f(x+2)$, and $2f(x+2)$. Determine which is the graph of $f(x)$ and match the other functions with their graphs.

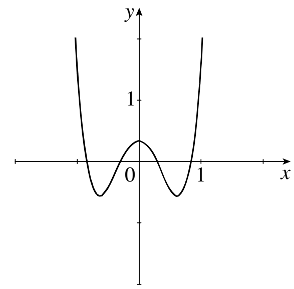
1.



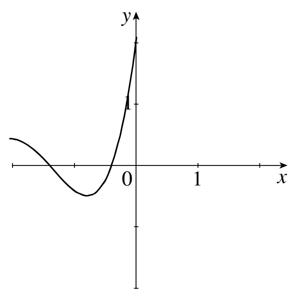
Graph 1



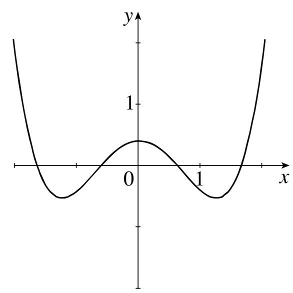
Graph 2



Graph 3

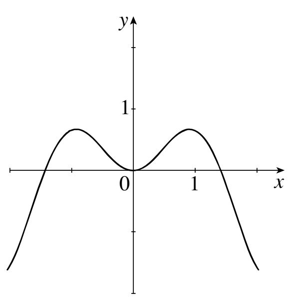


Graph 4

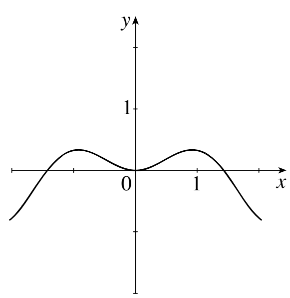


Graph 5

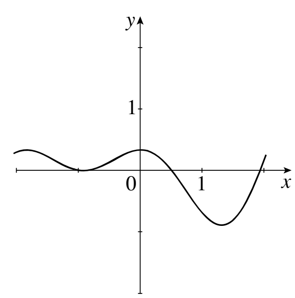
2.



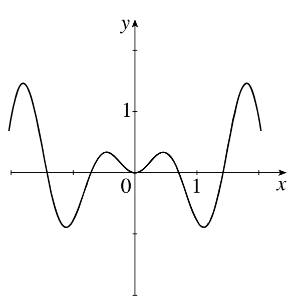
Graph 1



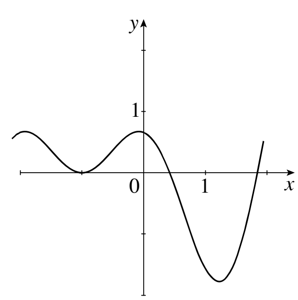
Graph 2



Graph 3



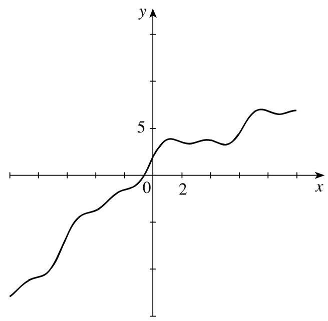
Graph 4



Graph 5

GROUP WORK 2, SECTION 1.3
Label Label Label, I Made it Out of Clay

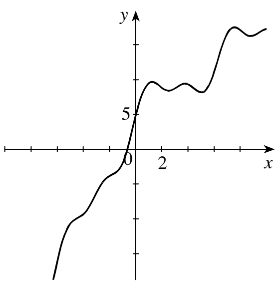
This is a graph of the function $f\,(x)$:



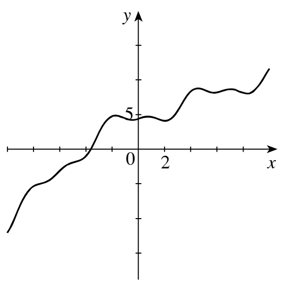
Give each graph below the correct label from the following:

(a) $f\,(x+3)$ (b) $f\,(x-3)$ (c) $f\,(2x)$ (d) $2f\,(x)$ (e) $|f\,(x)|$

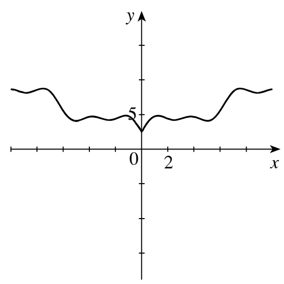
(f) $f\,(|x|)$ (g) $2f\,(x)-1$ (h) $f\,(2x)+2$ (i) $f\,(x)-x$ (j) $1/f\,(x)$



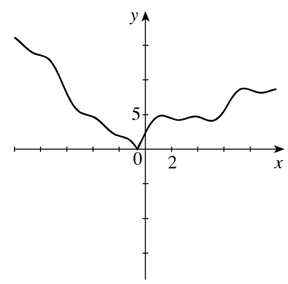
Graph 1



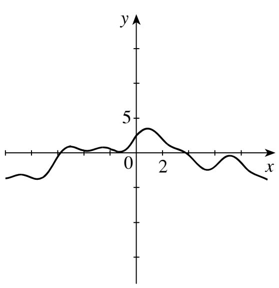
Graph 2



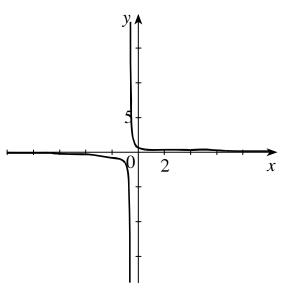
Graph 3



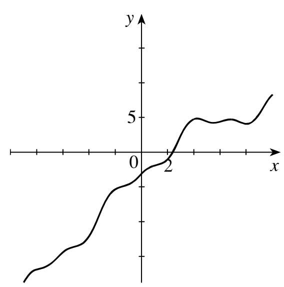
Graph 4



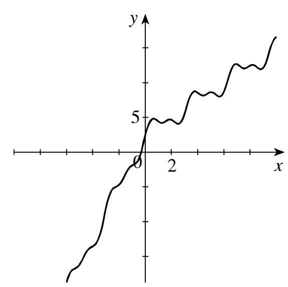
Graph 5



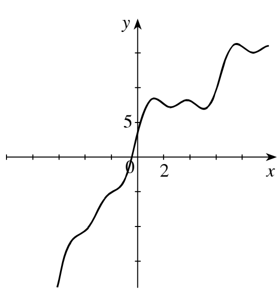
Graph 6



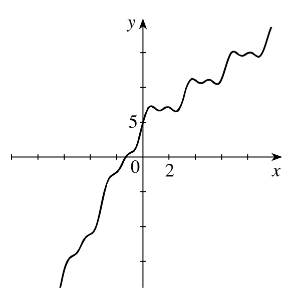
Graph 7



Graph 8



Graph 9



Graph 10

GROUP WORK 3, SECTION 1.3
It's More Fun to Compute

Using the graph below, find the following quantities.

1. $(f \circ g)(5)$

2. $(g \circ f)(5)$

3. $(f \circ g)(0)$

4. $(f \circ f)(5)$

5. $(g \circ g)(5)$

6. $(g \circ g)(-3)$

7. $(g \circ g)(-1)$

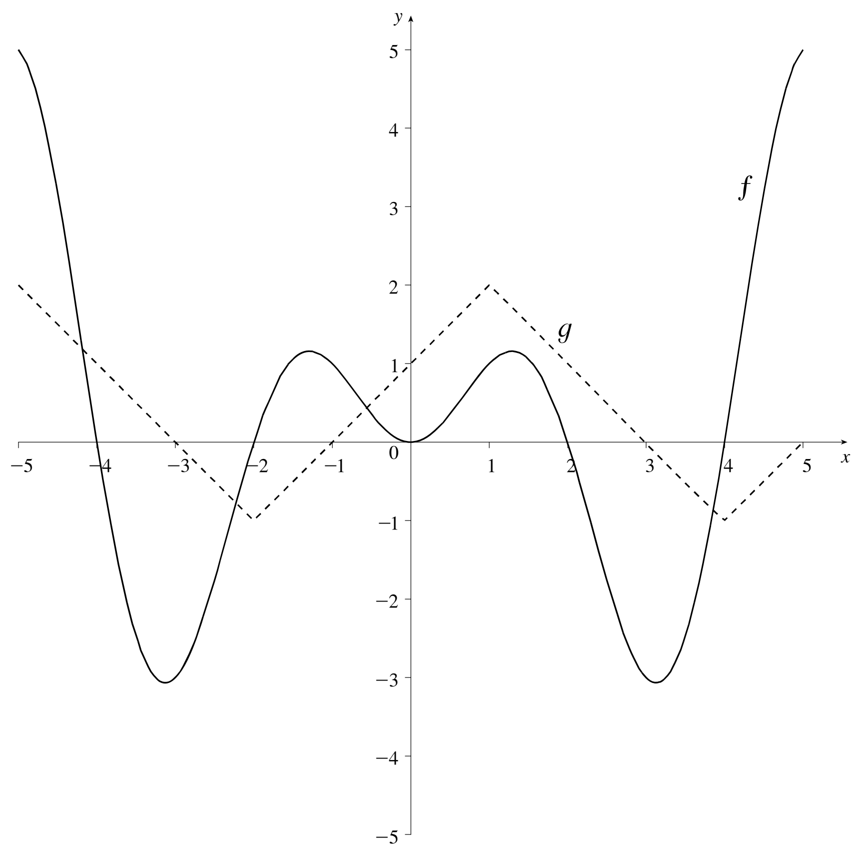
8. $(f \circ g)(1)$

9. $(g \circ f)(1)$

10. $(f \circ f \circ g)(4)$

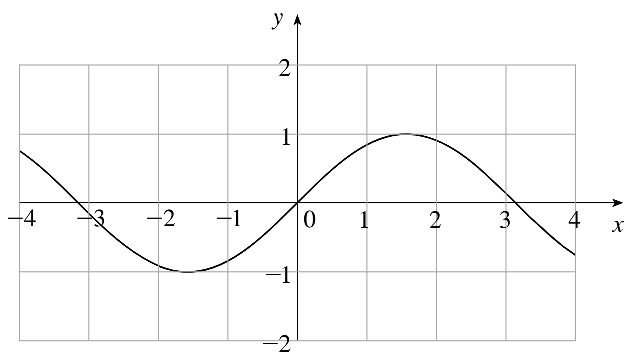
11. $(g \circ f \circ f)(4)$

12. $(f \circ g \circ f)(4)$



GROUP WORK 4, SECTION 1.3
Composing Transformations

The graph of $f(x) = \sin x$ is given below. For each of the transformed functions, explain in words how the graph can be obtained from the graph of $f(x)$, and sketch a graph of the transformed function. Then verify your sketch using technology.



1. $2\sin(x+3)$

2. $2\sin x + 3$

3. $\sin(2x) + 3$

4. $\sin(2x+3)$

5. $\sin(2(x+3))$

6. Questions 4 and 5 both involve a horizontal shift by 3 and a horizontal stretch by a factor of 2. But the graphs of $y = \sin(2x+3)$ and $y = \sin(2(x+3))$ are different! For which of the following pairs of transformations does the shape of the graph depend on the order in which the transformations are performed?

(a) Vertical stretch by a factor of 2, horizontal stretch by a factor of 3

(b) Vertical stretch by a factor of 2, horizontal shift by 3

(c) Horizontal stretch by a factor of 2, vertical shift by 3

(d) Horizontal shift by 2, vertical shift by 3

(e) Horizontal shift by 2, reflection about y-axis

1.4 Graphing Calculators and Computers

Suggested Time and Emphasis

$\frac{1}{2}$–1 class Optional material (If unassigned, students should be encouraged to read this self-contained section on their own.)

Points to Stress

1. When graphing an arbitrary function, some viewing windows are more appropriate than others, depending on the context of the inquiry and some analysis of the actual equation.
2. Some functions don't have any single viewing window that will give all the important details of the function.
3. One can use zoom and trace features to obtain estimates of solutions to difficult algebraic equations.
4. Graphing calculators can give misleading or wrong answers.

Text Discussion

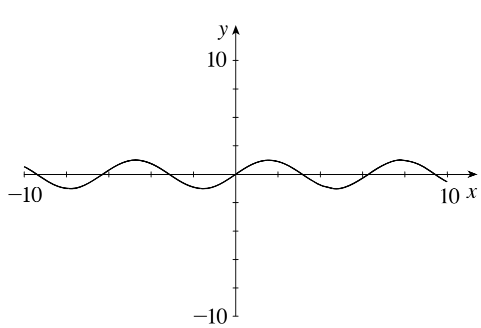
- Why is it true that "The solutions of the equation $\cos x = x$ are the x-coordinates of the points of intersection of the curves $y = \cos x$ and $y = x$"?

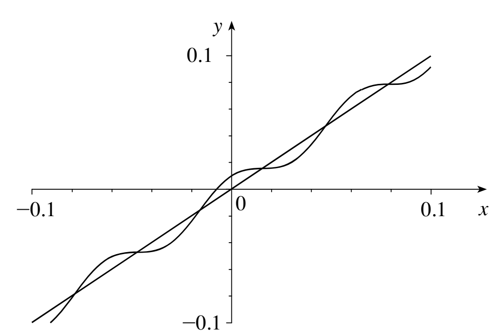
  ANSWER If $y = \cos x$ and $y = x$, we can use substitution to set $x = y = \cos x$.
- In Example 8, the text refers to "the number c". Is c really a number? Is c a variable?

Materials for Lecture

- Caution students to take care when using their calculators, particularly when choosing a viewing window. The following are features of a function that are difficult to determine solely from computer graphics:
  - End behavior. For example, $f(x) = \ln x$
  - Certain asymptotes, such as oblique asymptotes. For example, $f(x) = 3x + 2 + \dfrac{100 \sin x}{\ln x}$, $x > 2$
  - The effects of parameters. For example, $f(x) = \sin\left(Ax^b\right)$
  - Hidden roots. For example, the roots of $f(x) = x^4 - 0.001x$
  - Domains and ranges. For example, the domain of $\sqrt{\sin(x + \cos x)}$, the range of $\ln(\ln x)$
- Let $f(x) = \sin x + \frac{1}{100} \cos 100x$. In the standard viewing rectangle, the graph of $f(x)$ is indistinguishable from that of $\sin x$. Show that the viewing rectangle $[-0.1, 0.1] \times [-0.1, 0.1]$ reveals the "bumps" in the function.

  ANSWER





- Demonstrate the use of a calculator as an investigative tool (for example, to explore a family of functions like $f(x) = Qx + \cos x$ by graphing with $Q = 0.5$, 0.75, 0.95, 1, and 1.1). Discuss the meaning of "parameter" using Q as an example.
- Pose the question, "What happens to $\dfrac{\cos x - 1}{x}$ as x gets close to zero?" Investigate this question first numerically, and then by graphing.
  ANSWER Its value gets close to 0.
- Pose the question, "What are the roots of $f(x) = x^2 - x + 0.14$?" Here the graphing method of solution can be misleading, because of the calculator's resolution. Another dramatic illustration is trying to determine the long-term behavior of $\lim_{x\to\infty} \ln(\ln x)$ using the calculator (although this one is nice to save for a quiz).
- Attempt to graph $f(x) = \sqrt{x-2}\,\sqrt{x-4}$, and to find its domain. Certain packages or calculators (like the TI-89, Maple, and Mathematica) will graph it incorrectly, with domain $(-\infty, 2) \cup (4, \infty)$. Discuss why some software makes this error.
  ANSWER For $x < 2$, each of these algebra packages converts both multiplicands into imaginary numbers, multiplies them, and then converts the product back into a real number. Similarly, have the students attempt to graph $< \sqrt[3]{x-1}$.

Workshop/Discussion

- Have the students determine good viewing rectangles for $f(x) = 2x^2 - 35x - 16$ and for $g(x) = \dfrac{x}{x^2 + 11x + 1}$.
- Have the students try to figure out what the graph of $f(x) = [x + 0.05\cos(20x)]^2$ looks like near $x = 0$ by experimenting with viewing windows. Then have them try to explain why their picture makes sense.
- Have the class experiment with $f(x) = \cos\left(x + \dfrac{1}{x^2}\right)$ near $x = 0$, and also for large values of x.
- Have the students first guess the shape of $f(x) = \cos\left(2^{\sin x}\right)$, and then graph it. Ask how they should have known it would be periodic.
- Have the students first determine where $f(x) = x^3 - 50x^2$ is larger than x^2. Point out that it is easier to use some algebra to solve this problem than to use a calculator
  ANSWER $x > 51$

Group Work 1: Short- and Long-Term Behavior

Before handing this exercise out, show what is meant by "long-term behavior".

Part of the idea of this exercise is getting the students to explore and play with functions (even unfamiliar ones) on their calculators. Encourage them to try varying the functions on the worksheet, and see what happens.

ANSWERS

1. (a) $f(x) \to \infty$ as $x \to \infty$ (b) $x = 0$, $x \approx -0.877$

2. (a) $g(x) \to \infty$ as $x \to \infty$ (g is asymptotic to $y = 6x$) (b) None

3. (a) $h(x) \to \infty$ as $x \to \infty$ (b) $x = 0, \pm\sqrt{3}$

4. (a) $j(x) \to 1$ as $x \to \infty$ (b) None

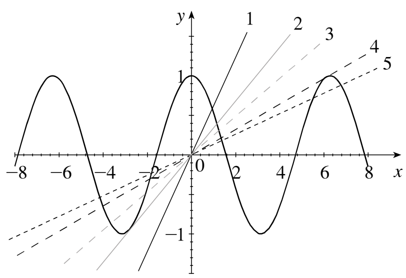
5. (a) $k(x) \to -\infty$ as $x \to \infty$ (b) $x \approx 0.4027, 0.9952, 1.923$

6. (a) $l(x) \to -\infty$ as $x \to \infty$ (b) $x = 0$

Group Work 2: Just Two Solutions

This is an extension of Exercise 24. It is a difficult exercise, and the students may require some guidance. Before handing out the hint sheet, have the students try to solve this problem on their own: "Consider the equation $\cos x = mx$. Find a value of m such that this equation has exactly two solutions." After they have worked on the problem, and are clear on what they are trying to find, hand out the hint sheet. Don't let them hang too long! Conclude by discussing how the line $y = mx$ that yields exactly two solutions is called tangent to the curve $y = \cos x$. Explain that soon we will learn how to find the equation of such a line. This is a good example to bring back once they have covered derivatives of trigonometric functions in Section 3.3. At that point, they will wind up having to solve the equation $x = -\cot x$. Revisit the problem again after covering Newton's method in Section 4.7, at which point they will finally have a way of approximating m to any desired accuracy.

ANSWERS 1, 3, 7.



2. 1, 3
5. $m \approx \pm 0.33651$
6. Symmetry

Homework Problems

CORE EXERCISES 1, 8, 13, 23, 25, 31

SAMPLE ASSIGNMENT 1, 2, 3, 5, 8, 11, 13, 15, 16, 18, 21, 23, 24, 25, 29, 31, 32, 35

EXERCISE	D	A	N	G
1				×
2				×
3				×
5				×
8				×
11				×
13				×
15				×
16				×

EXERCISE	D	A	N	G
18				×
21				×
23				×
24				×
25				×
29	×			×
31	×			
32	×			×
35	×			×

GROUP WORK 1, SECTION 1.4
Short- and Long-Term Behavior

For each of the following functions, (a) describe the long term behavior of the function, and (b) locate the zeros, if any.

1. $f(x) = \sin x + x^2$

2. $g(x) = 6x + \dfrac{1}{x} = \dfrac{6x^2 + 1}{x}$

3. $h(x) = \dfrac{x^3 - 3x}{x + 7}$

4. $j(x) = 2^{-1/x^6}$

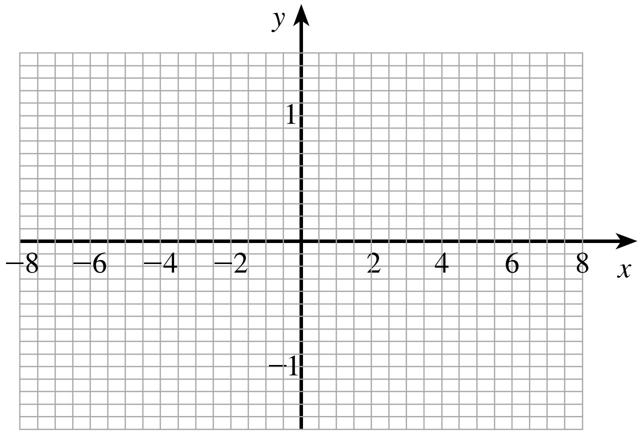
5. $k(x) = x^5 - 2x^4 - x^2 + 3x - 1.01^x$

6. $l(x) = Qx + \sin x$ for $Q = 0.32$, 0.9, and 1.1 What are the differences among these three functions?

GROUP WORK 2, SECTION 1.4
Just Two Solutions (Hint Sheet)

PROBLEM Find a value of m such that the equation $\cos x = mx$ has exactly two solutions.

1. Very carefully, draw graphs of the following three functions below: $y = \cos x$, $y = x$, $y = \frac{1}{5}x$.



2. From the graphs, determine how many solutions the equation has for $m = 1$ and for $m = \frac{1}{5}$.

3. Add a sketch of $y = mx$ onto your picture for Problem 1, where m is chosen so that the equation $\cos x = mx$ has exactly two solutions. You can do this without actually computing m.
4. Use your sketch to estimate the correct value of m.

5. Now use your graphing calculator to refine your guess. Try to get an answer correct to three decimal places.

6. Why is it true that the value $-m$ will also give exactly two solutions to $\cos x = mx$?

7. Is it possible to find a value of m such that $\cos x = mx$ has exactly three solutions? Four? n? Explain your answer by sketching $y = \cos x$ and $y = mx$ for the relevant values of m.

1.5 Exponential Functions

Suggested Time and Emphasis

$\frac{1}{2}$–1 class Essential or optional material, depending on student background

Points to Stress

1. Algebraic and geometric properties of exponential functions.

2. Growth rates of exponential functions as compared to polynomials.

3. Translation and reflection of exponential functions, from both symbolic and geometric perspectives.

4. Exponential functions as models for population growth and decay.

Text Discussion

- The half-life of Strontium 90 is 25 years. If you have 800 mg of Strontium 90, how much will be left after 100 years? Try to answer without reaching for a pencil or calculator.
  ANSWER About 50 mg

- $3^3 = 3 \cdot 3 \cdot 3$, $3^{3/4} = \sqrt[4]{3 \cdot 3 \cdot 3}$. How does one make sense of $3^{\sqrt{7}}$?
  ANSWER Answers will vary. One example: we can approximate $\sqrt{7}$ by a sequence of rational numbers, and can thus similarly approximate $3^{\sqrt{7}}$ to any degree of precision.

- If your wealth was e^t dollars after t years, after how many years would you become a billionaire?
  ANSWER $\ln(1000000000) \approx 20.723$ years

Materials for Lecture

- Start to draw a graph of 2^x vs x, using the scale of one inch per unit on both axes. Point out that after one foot, the height would be over 100 yards (the length of a football field). After two feet, the height would be 264 miles, after three feet it would be $1{,}000{,}000$ miles (four times the distance to the moon), after three and a half feet it would be in the heart of the sun. If the graph extended five feet to the right, $x = 60$, then y would be over one light year up.

- Discuss Exercise 6.

- Point out this contrast between exponential and linear functions: For equally spaced x-values, linear functions have constant *differences* in y-values, while pure exponential functions have constant *ratios* in y-values. Use this fact to show that the following table describes an exponential function, not a linear one.

x	y
-6.2	0.62000
-2.4	0.65100
1.4	0.68355
5.2	0.71773
9.0	0.75361
12.8	0.79129

- In 1985 there were 15,948 diagnosed cases of AIDS in the United States. In 1990 there were 156,024. Scientists said that if there was no research done, the disease would grow exponentially. Compute the number of cases this model predicts for the year 2000. The actual number was 774,467. Discuss possible flaws in the model with the students, and point out the dangers of extrapolation.

Workshop/Discussion

- Estimate where $3^x > x^3$ and where $2^x > x^8$ using technology. Notice that exponential functions start by growing *slower* than polynomial functions, and then wind up growing much *faster*. For example, if one were to graph x^2 vs x using one inch per unit, then when $x = 60$, y would be only 100 yards, as opposed to a light year for $y = 2^x$. (The sun is only 8 light minutes from the earth.)
  ANSWER $x > 1.175\,136$, $x > 1.1$
- TEC To further study the relationship between the base of an exponential function $f(x) = a^x$ and the shape of its graph, have the students do Exercises 1–4 in TEC Module 1.5.
- Estimate where $3^x > 10^7$ by use of technology. Then use algebra to find an exact answer. (Logarithms are briefly reviewed in Section 1.6.)
  ANSWER $x > \log_3(10^7)$ or $x > \dfrac{7 \ln 10}{\ln 3}$

Group Work 1: I've Grown Accustomed to Your Growth

Before handing out this activity, it may be prudent to review the rules of exponentiation. This exercise enables students to discover for themselves the equal ratio property of exponential functions.

ANSWERS 1. Yes ($m = 1$), no, yes ($m \approx 2.08$), yes ($m \approx 2.01$) **2.**Equally spaced changes in x-values result in equally spaced changes in y-values **3.** Equally spaced changes in x values result in equally proportioned changes in y-values with the same ratio. $b = 2$, $b = 0.9975$, $b = 2.25$, $b = 3$ **4.** The "$+C$" gets in the way when taking the ratio. However, the property is close to being true when A and b are large compared to C.

Group Work 2: Comparisons

The purpose of this group work is to give the students a bit of "picture sense." It is acceptable if they do this by looking at graphs on their calculators, setting the windows appropriately.

ANSWERS 1. $0 < x < 1.374$ and $x > 9.940$ **2.** $0 < x < 1.052$ and $x > 95.7168$ **3.** $0 < x < 1.240$ and $x > 16$ **4.** $0 < x < 1.517$ and $x > 7.175$

Group Work 3: Estimating e

This activity requires technology (such as TEC Module 1.5) that can be used to quickly compute the slope of a line tangent to the graph of a function. Alternatively, the difference quotient method of approximating a derivative near $x = 0$ can be shown to an advanced class at this time.

ANSWERS **1.** The slopes are approximately 0.956, 1.030 and 1.099. They increase as a increases. **2.** 2.6 and 2.8 **3.** 0.956, 0.993, 1.030; between 2.7 and 2.8 **4.** $e \approx 2.7183$

Homework Problems

CORE EXERCISES 3, 4, 5, 9, 13, 19, 21, 26, 30

SAMPLE ASSIGNMENT 2, 3, 4, 5, 9, 13, 14, 17, 19, 20, 21, 23, 24, 26, 30, 31, 36

EXERCISE	D	A	N	G
2		×		
3		×		
4		×		
5	×	×		×
9	×			×
13				×
14				×
17				×
19		×		
20		×		×
21		×	×	×
23		×		
24	×	×		
26			×	×
30			×	×
31		×	×	
36	×			×

GROUP WORK 1, SECTION 1.5
I've Grown Accustomed to Your Growth

1. Some of the following four tables of data have something in common: linear growth. Without trying to find complete equations of lines, determine which of them are linear growth, and determine their rate of change:

x	y
1	2
2	3
3	4
4	5

x	y
21.5	4.32
32.6	4.203
43.7	4.090
54.8	3.980

x	y
-3	1.1
-2.5	2.14
-2	3.18
-1.5	4.22

x	y
1	-5.00
3	-0.98
6	5.05
8	9.07

2. In a sentence, describe a property of linear growth that can be determined from a table of values.

3. The following four tables of data have something in common: exponential growth. Functions of the form $y = Ab^x$ (or Ae^{kx}) have a property in common analogous to the one you stated in Question 2. Find the property, and then find the value of b.

x	y
1	5
2	10
3	20
4	40

x	y
21.5	4.32
32.6	4.203
43.7	4.090
54.8	3.980

x	y
-3	1.1
-2.5	1.65
-2	2.475
-1.5	3.7125

x	y
1	0.8
3	7.2
6	194.4
8	1749.6

4. Unfortunately, the above property does not hold for functions of the form $y = Ab^x + C$. What goes wrong? For what kinds of values of A, b, and C does the property come close to being true?

GROUP WORK 2, SECTION 1.5
Comparisons

You have learned that an exponential function grows faster than a polynomial function. Find the values of $x > 0$ for which

1. $2^x \geq x^3$

2. $(1.1)^x \geq x^2$

3. $2^x \geq x^4$

4. $3^x \geq x^4$

GROUP WORK 3, SECTION 1.5
Estimating e

One definition of e is as the number a such that the slope of the tangent line to the graph of $y = a^x$ at $x = 0$ is 1.

1. Use technology to compute the slope of the line tangent to the graph of $y = a^x$ at $x = 0$ for $a = 2.6, 2.8$, and 3.0. Does it seem that the slope increases as a increases?

2. Based on the results of Part 1, and keeping in mind the definition given above, would you guess that e is between 2.6 and 2.8 or between 2.8 and 3.0? Why?

3. Compute the slope of the tangent line to the graph of $y = a^x$ at $x = 0$ for $a = 2.6, 2.7$, and 2.8. Would you guess that e is between 2.6 and 2.7, or between 2.7 and 2.8?

4. Repeat Part 3, each time narrowing the range for a, until you have an estimate of e you are confident is correct to three decimal places.

1.6 Inverse Functions and Logarithms

Suggested Time and Emphasis

1 class Essential material

Points to Stress

1. The use of multiple representations (verbal, numeric, visual, algebraic) to understand inverse functions, always coming back to the central idea of reversing inputs and outputs.
2. Logarithmic functions and their algebraic and geometric properties.
3. Tests for one-to-one functions.

Text Discussion

- In this section, the author describes a technique to graph the inverse of a one-to-one function using the line $y = x$. Why does this technique work? Does it work for $y = x^2$?

  ANSWER It works because reflecting across the line $y = x$ is the same as reversing the inputs and outputs of a function. It doesn't work for $y = x^2$ because $y = x^2$ is not one-to-one.
- What is the inverse function f^{-1} for $f(x) = 2(\log_3 x + 4)$?

  ANSWER $f^{-1}(x) = 3^{(x/2)-4}$

Materials for Lecture

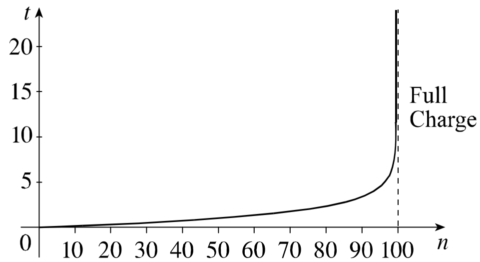
- Sketch a graph of $f(x) = 4\ln(x + 2)$, starting with the graph of $y = \ln x$. Then show that this is also a graph of the function written as $\ln(x + 2)^4$.
- When the logarithm function is graphed on a calculator, it appears to have a horizontal asymptote. Point out that the graph is misleading in that way. Start a graph of $y = \log_{10} x$ on the blackboard, noting the domain and the vertical asymptote. Using the scale of 1 inch $= 1$ unit (on the x-axis) and 1 foot $= 1$ unit (on the y-axis), plot some points:

x	0.1	1	2	3	4	5	6	7	8	9	10
$\log_{10} x$	-1	0	0.30	0.47	0.60	0.70	0.78	0.85	0.90	0.95	1

  Now ask how many inches we would have to go out to get up to $y = 2$ feet. (Answer: 100 inches, or $8\frac{1}{3}$ feet.) If the blackboard is large enough, plot the point $(100, 2)$. Then ask how far we would have to go to get up to $y = 5$ feet. (Answer: 1.57 miles.) Note how it turns out to take close to a mile and a half to go from $y = 4$ to $y = 5$, and that (if you graphed it out) it would look a lot like there is a horizontal asymptote. Find the distance from your classroom to a city or landmark in another state, and ask the class to estimate the log of that distance, using the same scale.

Workshop/Discussion

- Ask your students if they have ever had to deal with recharging a battery, for example the battery to a music device or a laptop. Assume that the battery is dead, and it takes an hour to charge it up half way. Ask them how much it will be charged in two hours. (The answer is 75% charged.) They may have noticed that, when charging a laptop battery, it takes a surprisingly long time for the monitor to change from "99% charged" to "100% charged." It turns out that the time it takes to charge the battery to $n\%$ is given by $t = -k \ln\left(1 - \frac{n}{100}\right)$; in our example $k = 1.4427$. Have students compute how long it would take the battery to get a 97% charge, a 98% charge, and a 99% charge. (Remind them that it took only an hour to go from 0% to 50%.) Graph t versus n to demonstrate that the battery will never be fully charged.

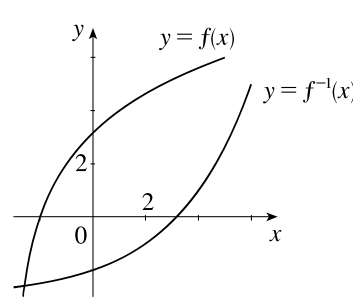


- Show students semilog graph paper (available at university bookstores, from your friendly neighborhood physics teacher, or from websites such as `http://tinyurl.com/2qqbyb`.) Point out how the distance between the y-axis lines is based on the logarithm of the y-coordinate, not on the y-coordinate itself. Have them graph $y = 2^x$ on semilog graph paper.
- Use graphing technology to check if $f(x) = x^3 - 2x^2 - x + 2$ is one-to-one.

  ANSWER It is not. It fails the Horizontal Line Test.
- Starting with $f(x) = \sqrt[3]{x-4}$ compute $f^{-1}(-2)$ and $f^{-1}(0)$. Then use algebra to find a formula for $f^{-1}(x)$. Have the students try to repeat the process with $g(x) = x^3 + x - 2$. Note that facts such as $g^{-1}(-2) = 0$, $g^{-1}(0) = 1$, and and $g^{-1}(8) = 2$ can be found by looking at a table of values for $g(x)$ but that the algebraic approach fails to give us a general formula for $g^{-1}(x)$. Finally, draw graphs of f, f^{-1}, g, and g^{-1}.
- Find the inverse of $f(x) = \sqrt{4x+3}$ algebraically and graphically by reflecting about the line $y = x$. Discuss the domain and range of $f^{-1}(x)$.
- Pose the question: If f is always increasing, is f^{-1} always increasing? Give the students time to try prove their answer.

  ANSWER This is true. Proofs may involve diagrams and reflections about $y = x$, or you may try to get them to be more rigorous. This is an excellent opportunity to discuss concavity, noting that if f is concave up and increasing, then f^{-1} is concave down and increasing.
- Sketch a graph of $f(x) = \log_2\left((x+3)^2\right)$, sketch the inverse function, and then find an algebraic formula for the inverse.

  ANSWER



$f^{-1}(x) = 2^{x/2} - 3$

Group Work 1: Inverse Functions: Domains and Ranges

While discussing the domains and ranges of inverse functions, this exercise will also foreshadow later excursions into the maximum and minimum values of functions.

If a group finishes early, ask them this question:

"Now consider the graph of $f(x) = \sqrt{2x-3}+2$. What are the domain and range of $f(x)$? Try to figure out the domain and range of $f^{-1}(x)$ by looking at the graph of f. In general, what information do you need to be able to compute the domain and range of $f^{-1}(x)$ from the graph of a function f?"

ANSWERS

1. It is one-to-one, because the problem says it climbs steadily.
2. a^{-1} is the time in minutes at which it achieves a given altitude.
3. Reverse the data columns in the given table to get the table for the inverse function. The domain and range of a are $0 \leq t \leq 30$ and $0 \leq a \leq 29{,}000$, so the domain and range of a^{-1} are $0 \leq x \leq 29{,}000$ and $0 \leq a^{-1} \leq 30$.
4. After approximately 8.5 minutes
5. a is no longer 1-1, because heights are now achieved more than once.

BONUS The domain of f^{-1} is the set of all y-values on the graph of f, and the range of f^{-1} is the set of all x-values on the graph of f.

Group Work 2: Functions in the Classroom

Before starting this one, review the definition of "function". Some of the problems can only be answered by polling the class after they are finished working. Don't forget to take leap years into account for the eighth problem. For an advanced class, follow up by defining "one-to-one" and "bijection", then determining which of the functions have these properties.

ANSWERS

Chairs: Function, one-to-one, bijection (if all chairs are occupied)

Eye color: Function, not one-to-one

Mom & Dad's birthplace: Not a function; mom and dad could have been born in different places

Molecules: Function, one-to-one (with nearly 100% probability); inverse assigns a number of molecules to the appropriate student.

Spleens: Function, one-to-one, bijection. Inverse assigns each spleen to its owner.

Pencils: Not a function; some people may have more than one or (horrors!) none.

Student number: Function, one-to-one; inverse assigns each number to its owner.

February birthday: Not a function; not defined for someone born on February 29.

Birthday: Function, perhaps one-to-one.

Cars: Not a function; some have none, some have more than one.

Cash: Function, perhaps one-to-one.

Middle names: Not a function; some have none, some have more than one.

Identity: Function, one-to-one, bijection. Inverse is the same as the function.

Calculus instructor: Function, not one-to-one.

Number of hairs: Function, not one-to-one. There are more people in New York City than there are possible values for this function. Therefore, at least two New Yorkers have the same number of hairs on their heads, and so the function does not have an inverse.

Group Work 3: Irrational, Impossible Relations

Before starting this activity, review the definitions of rational and irrational numbers. The hint sheet should be given out only after the students have tried to show that $\log_2 3$ is irrational, or at least discussed it enough to understand what they are trying to show. If a group finishes early, have them show that $\log_2 a$ is always irrational if a is an odd integer.

Group Work 4: The Column of Liquid

If the students need a hint, you can mention that the liquid used in the drugstore is mercury.

ANSWERS

1. The liquid is 1 cm high when the temperature is 32 °F.
2. The liquid is 2 cm high when the temperature is 212 °F
3. The inverse function takes a height in cm, and gives the temperature. So it is a device for measuring temperature.
4. A thermometer

Homework Problems

CORE EXERCISES 3, 13, 18, 19, 25, 31, 37, 52

SAMPLE ASSIGNMENT 3, 6, 9, 10, 13, 15, 18, 19, 22, 25, 26, 28, 29, 31, 32, 35, 36, 37, 39, 50, 52, 55, 59, 61

EXERCISE	D	A	N	G
3			×	
6				×
9		×		
10		×		
13	×			
15		×	×	
18			×	×
19	×	×		
22		×		
25		×		
26		×		
28		×		
29				×
31	×	×		×
32	×			×
35			×	
36			×	
37			×	
39		×		
50		×		
52		×		
55		×		
59	×	×	×	
61		×		×

GROUP WORK 1, SECTION 1.6
Inverse Functions: Domains and Ranges

Let $a(t)$ be the altitude in feet of a plane that climbs steadily from takeoff until it reaches its cruising altitude after 35 minutes. We don't have a nice formula for a, but extensive research has given us the following table of values:

t	$a(t)$
0.1	50
0.5	150
1	500
3	2000
7	8000
10	12,000
20	21,000
25	27,000
30	28,500
35	29,000

1. Is $a(t)$ a one-to-one function on the interval $[0, 35]$? How do you know?

2. What does the function a^{-1} measure in real terms? Your answer should be descriptive, similar to the way $a(t)$ was described above.

3. We are interested in computing values of a^{-1}. Fill in the following table for as many values of x as you can. What quantity does x represent?

x	$a^{-1}(x)$

What are the domain and range of a? What are the domain and range of a^{-1}?

4. You are allowed to turn on electronic equipment after the plane has reached 10,000 feet. Approximately when can you expect to turn on your laptop computer after taking off?

5. Suppose we consider $a(t)$ from the time of takeoff to touchdown. Is $a(t)$ still one-to-one?

GROUP WORK 2, SECTION 1.6
Functions in the Classroom

Which of the following relations are functions?

Domain	Function Values	Function
All the people in your classroom	Chairs	f (person) = his or her chair
All the people in your classroom	{blue, brown, green, hazel}	f (person) = his or her eye color
All the people in your classroom	Cities	f (person) = birthplace of his or her mom and dad
All the people in your classroom	$\mathbb{R}$, the real numbers	f (person) = number of molecules in his or her body
All the people in your classroom	Spleens	f (person) = his or her spleen
All the people in your classroom	Pencils	f (person) = his or her pencil
All the students in your classroom	Integers from 0–99999999	f (person) = his or her student number
All the living people born in February	Days in February, 2007	f (person) = his or her birthday in February 2007
All the people in your classroom	Days of the year	f (person) = his or her birthday
All the people in your classroom	Cars	f (person) = his or her car
All the people in your classroom	$\mathbb{R}$, the real numbers	f (person) = how much cash they have on them
All the people in your college	Names	f (person) = his or her middle name
All the people in your classroom	People	f (person) = theirself
All the students in your classroom	People	f (person) = his or her calculus instructor
All the people in New York City	$\mathbb{W}$, the whole numbers	f (person) = the number of hairs on his or her head

GROUP WORK 3, SECTION 1.6
Irrational, Impossible Relations

1. If $\log_2 x = s$, then what is $\log_{1/2} x$?

2. If $\log_b x = s$, then what is $\log_{1/b} x$ (assuming $b > 1$)?

3. If $\log_b x = s$, then what is $\log_{b^2} x$?

We are going to estimate $\log_2 3$. In pre-calculus, you memorized that $\log_2 3 \approx 1.584962501$. Suppose you didn't have this fact memorized. There is no $\log_2$ button on your calculator! How would you compute it?

Unfortunately, the calculator gives us only a finite number of digits. If $\log_2 3$ were a rational number, we would be able to express it as a fraction, giving us perfect accuracy. Do you think it is rational or irrational? Try to prove your result.

GROUP WORK 3, SECTION 1.6
Irrational, Impossible Relations (Hint Sheet)

So, you realize that it's not easy to determine whether $\log_2 3$ is rational!

One way to attempt to show that $\log_2 3$ is rational is to assume that it is, and try to find integers a and b such that $\log_2 3 = \frac{a}{b}$. If we can show that there are no such a and b, then $\log_2 3$ *cannot* be rational.

1. Assume that $\log_2 3 = \frac{a}{b}$ for integers $a, b \geq 0$. Show that a and b must then satisfy $2^a = 3^b$.

2. Notice that $a = 0$, $b = 0$ satisfies $2^a = 3^b$. Why doesn't this fact help us?

3. Find $a \neq 0$ and $b \neq 0$ that satisfy $2^a = 3^b$, or show that no such a and b exist.

4. Is $\log_2 3$ rational or irrational? Why?

GROUP WORK 4, SECTION 1.6
The Column of Liquid

It is a fact that if you take a tube and fill it partway with liquid, the liquid will rise and fall based on the temperature. Assume that we have a tube of liquid, and we have a function $h(T)$, where h is the height of the liquid in cm at temperature T in $^\circ$F.

1. It is true that $h(32) = 1$. What does that mean in physical terms?

2. It is true that $h(212) = 10$. What does that mean in physical terms?

3. Describe the inverse function h^{-1}. What are its inputs? What are its outputs? What does it measure?

4. There is a device, currently available at your local drugstore, that measures the function h^{-1}. What is the name of this device?

1.7 Parametric Curves

NOTE This material is used in Section 3.4 (to find tangents to parametric curves), Section 4.4 (to graph parametric curves), Section 6.1 (to find areas bounded by parametric curves), and Section 6.4 (to find lengths of parametric curves).

Suggested Time and Emphasis

1 class Essential material

Points to Stress

1. The definition of parametric equations.

2. Relations generated by parametric equations, some of which do not describe functions.

3. Sketching parametric curves.

4. Using parametric equations to describe inverse functions.

Text Discussion

- What is the difference between a function and a parametric curve?
  ANSWER Many answers are possible. The graph of a function can be made into a parametric curve, but not necessarily the other way around. A function has to pass the vertical line test and a parametric curve does not.

- Can different parametric equations represent the same curve?

- Can you easily find an equation for the inverse f^{-1} of $f(x) = \sqrt{x^3 + x^2 + x + 1}$, and graph it without using parametric equations?

Materials for Lecture

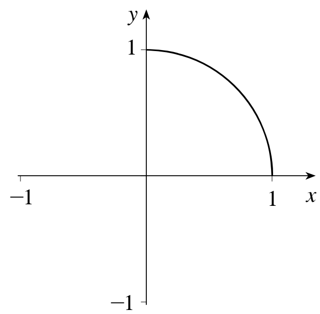
- Revisit Examples 2 and 3 using different parametric representations and speeds, such as $x(t) = \sin\left(e^t\right)$, $y(t) = \cos\left(e^t\right)$, $\ln\pi \leq t \leq \ln 3\pi$.

- Caution students to take appropriate care in sketching parametric curves, especially concerning questions of range and direction.

- Describe the difference between a function curve and a parametric curve, perhaps using Example 5. Caution students to take appropriate care in sketching parametric curves, especially concerning questions of range and direction.

- Describe the graph of the parametric equations $x(t) = |\sin t|$, $y(t) = |\cos t|$.

  ANSWER



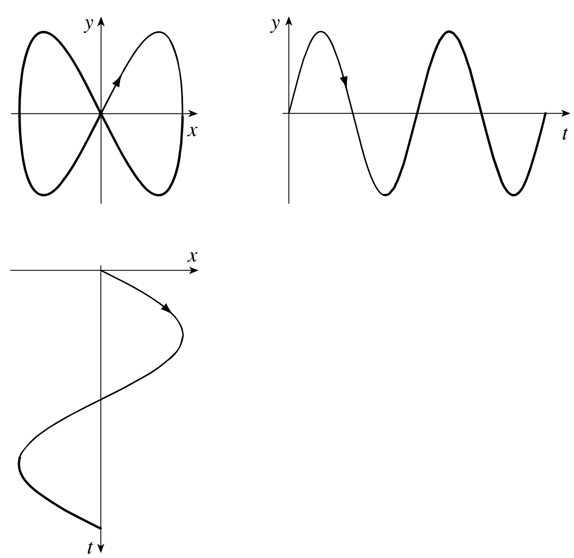
- Describe two different parametrizations for the ellipse $\frac{x^2}{9} + \frac{y^2}{4} = 1$

  ANSWER Examples are $x(t) = 3\cos t$, $y(t) = 2\sin t$ and $x(t) = 3\cos 4t$, $y(t) = 2\sin 4t$.

- Display an inverse for $f(x) = x^3 + x + 2$ graphically using parametric equations. Explain the difficulties with the algebraic approach.

- Discuss the process of going from a parametric curve to a relation between x and y. The figure at right is meant to help students see the way that parametrized curves are sketched out over time. First sketch $(x(t), y(t))$, starting at the initial point (the origin), and moving up and to the right. (Try to keep your speed constant.) Stop when the cycle is about to repeat. Then, to the right of the figure, graph the motion in the y-direction only. Then, below the figure, graph the motion in the x-direction. That graph is sideways because the x-axis is horizontal.

  This process can be viewed for trigonometric functions and cubic polynomials using TEC.



- Discuss the cycloid (Example 7) using TEC to illustrate.

Workshop/Discussion

- Have the students sketch the following curves using the parametric equations, and then eliminate the parameters to find Cartesian equations. Visualization using TEC could be helpful.

  (a) $x(t) = \frac{1}{2}t^3 - 2t$, $y(t) = \frac{1}{2}t^3 - 2t$, $0 \leq t \leq 3$

  (b) $x(t) = \frac{1}{2}t^3$, $y(t) = \frac{1}{2}t^3 - 2t$, $0 \leq t \leq 3$

- Using parametric equations, have students graph the inverse function of $y = x^3 + x$.
- Ask the students to describe the graph of the parametric equations $x = a\cos t$, $y = b\sin t$ for various values of a and b, such as
  (a) $a = 3, b = 3$ (b) $a = 2, b = 2$ (c) $a = 1, b = 3$ (d) $a = 3, b = 1$ (e) $a = 6, b = 2$
- Have the students guess several different parametrizations for the ellipse $9x^2 + 16y^2 = 25$. Students can visualize guesses involving combinations of sines and cosines using TEC.
- Have the students try to guess the shape of $x(t) = 2\cos 4t$, $y(t) = 3\sin 4t$ by first looking at the individual graphs of $x(t)$ and $y(t)$. Then repeat this process using $x(t) = 2\cos 4t$, $y(t) = 3\sin 5t$. If they follow well, try $x(t) = 2\cos 4t$, $y(t) = 3\sin\sqrt{15}\,t$.
- Give an example of a "space-filling" curve such as $x(t) = \cos(et)$, $y(t) = \sin\left(\sqrt{3}\,t\right)$. This curve essentially fills the square $-1 \leq x \leq 1$, $-1 \leq y \leq 1$ in that the curve gets arbitrarily close to any point in the square. It can be simulated using TEC or a graphing calculator with the approximations $e \approx 2.7183$, $\sqrt{3} \approx 1.7321$. The range $0 \leq t \leq 200$ should be sufficient to approximately fill most of the square.

Group Work 1: Name that Parametrization

This exercise gives the students some practice playing with parametric curves. Before starting the activity, make sure that the students know how to graph a simple set of parametric curves using their technology.

ANSWERS

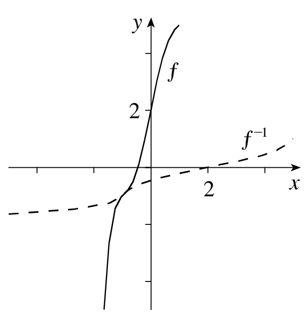
1. $y = \left(1 - x^2\right) - x^2 = 1 - 2x^2$

3. They are the same if considered as curves in the plane. The second one "moves" twice as fast as the first.

4. The curves are inverses of each other (or, they are symmetric about the line $y = x$).

5. (a) Use the technique of Exercise 4.

(b)



Group Work 2: How Many Ways Can You Trace That Curve?

This exercise gives students practice in finding different parametrizations for the circle $x^2 + y^2 = 4$, using the forms $x(t) = a_1 \cos a_2 t$, $y(t) = a_3 \sin a_4 t$. Students may use a graphing calculator or TEC.

ANSWERS

1. Each of a_1 and a_3 must be ± 2.

2. Once around counterclockwise, starting at $(1, 0)$.

3. $a_2 = a_4 = 2$, $a_2 = a_4 = 3$

4. Once around clockwise, starting at $(1, 0)$.

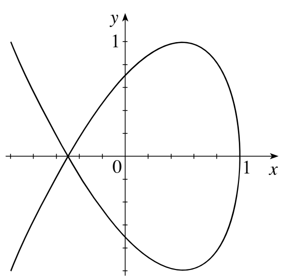
5. $a_2 = 3$, $a_4 = -3$; $a_2 = 5$, $a_4 = -5$

TEC Group Work 3: Lissajous Figures

This activity is an extension of Exercise 43. Students can use a graphing calculator or TEC to generate the figures.

ANSWERS

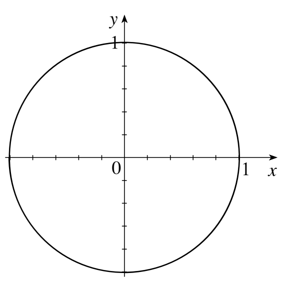
1.



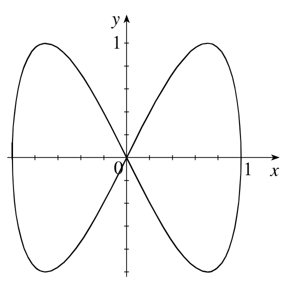
$a_2 = 2$, $a_4 = 3$

All the graphs look the same. The speed at which they are traced out varies. If a_1 and a_3 are fixed, then the graph is determined by the ratio of a_2 to a_4.

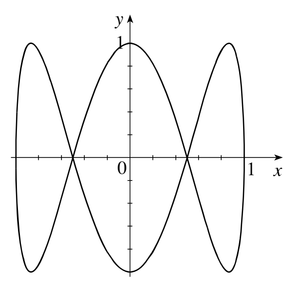
2.



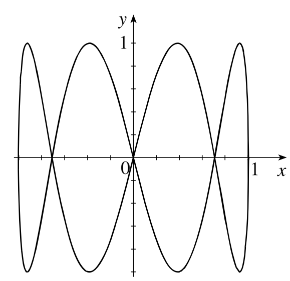
$a_2 = 1$, $a_4 = 1$



$a_2 = 1$, $a_4 = 2$



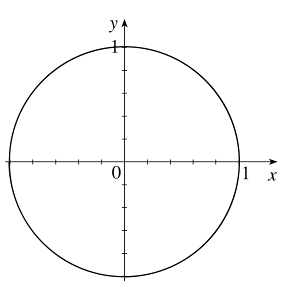
$a_2 = 1$, $a_4 = 3$



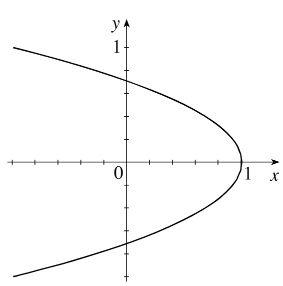
$a_2 = 1$, $a_4 = 4$

As a_4 increases, the figure doubles back on itself more in the vertical direction.

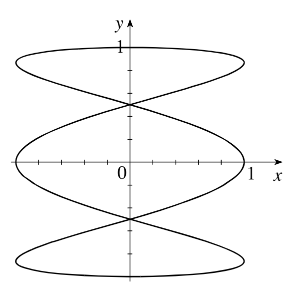
3.



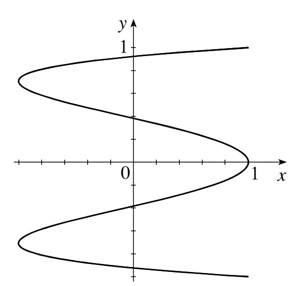
$a_2 = 1$, $a_4 = 1$



$a_2 = 2$, $a_4 = 1$



$a_2 = 3$, $a_4 = 1$

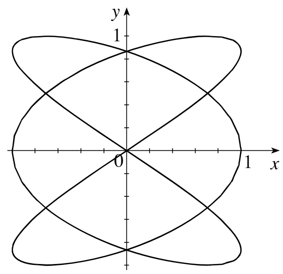


$a_2 = 4$, $a_4 = 1$

The behavior is different, but analogous. This time, the doubling back doesn't always yield a new loop.

4. It is a parabola. This is because $x = \cos 2t = \cos^2 t - \sin^2 t = 1 - 2\sin^2 t = 1 - 2y^2$.

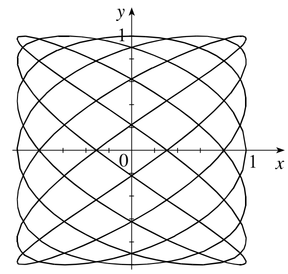
5.



$a_2 = 3, a_4 = 2$

$b = 2\pi$. x goes through three cycles, and y goes through two. This can be seen by tracing the graph with a finger, paying attention first to the cycles in the x-direction, then to the cycles in the y-direction.

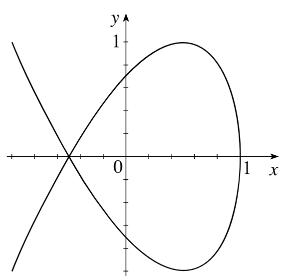
6.



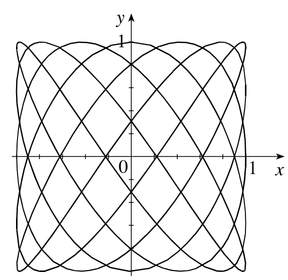
$a_2 = 7, a_4 = 5$

$b = 2\pi$. x goes through seven cycles, and y goes through five. This can be seen by tracing the graph with a finger, paying attention first to the cycles in the x-direction, then to the cycles in the y-direction.

7.



$a_2 = 2, a_4 = 3$



$a_2 = 5, a_4 = 7$

There are several reasons the graphs look different. One reason is that $\cos(2\pi - x) = \cos x$ while $\sin(\pi - x) = \sin x$, so the functions have different symmetries on the interval $0 < t < 2\pi$.

Homework Problems

CORE EXERCISES 1, 3, 5, 10, 17, 23, 26, 31, 36, 42

SAMPLE ASSIGNMENT 1, 3, 5, 7, 10, 14, 17, 23, 26, 31, 36, 42

EXERCISE	D	A	N	G
1				×
3				×
5		×		×
7		×		×
10		×		×
14		×		×
17	×	×		×
23				×
26	×			×
31		×		
36	×			×
42	×	×	×	×

GROUP WORK 1, SECTION 1.7
Name that Parametrization

1. Consider the graph of the following set of parametric equations:

$$x(t) = \sin t \qquad y(t) = \cos 2t \qquad 0 < t < \infty$$

Graph this curve with your calculator, then write the equations in the form $y = f(x)$. (*Hint:* Use the formula $\cos 2\theta = \cos^2\theta - \sin^2\theta$.)

2. Try to guess what the graph of the following set of parametric equations looks like, and then see if you are right.

$$x(t) = \sin 2t \qquad y(t) = \cos 6t \qquad 0 \le t \le 4\pi$$

These curves are called Lissajous figures, and are used in electrical engineering to see if two signals are "in sync".

3. Consider these two sets of parametric equations:

$$x(t) = t \qquad y(t) = \sin t \qquad 0 < t < \infty$$

$$x(t) = 2t \qquad y(t) = \sin 2t \qquad 0 < t < \infty$$

What is the relationship between their associated curves?

4. Given any set of equations of the form

$$x(t) = t \qquad y(t) = f(t) \qquad 0 < t < \infty$$

What does the graph of the set of equations

$$y(t) = t \qquad x(t) = f(t) \qquad 0 < t < \infty$$

look like?

5. (a) Use a graphing calculator to check that $f(x) = x^5 - 3x^3 + 5x + 2$ is one-to-one.

(b) Graph its inverse function f^{-1}.

GROUP WORK 2, SECTION 1.7
How Many Ways Can You Trace That Curve?

Consider the circle $x^2+y^2=4$. We want to construct parametric curves $x(t)=a_1\cos a_2t$, $y(t)=a_3\sin a_4t$, with a_1, $a_2>0$, which will trace this circle in different ways.

1. What must be the values of a_1 and a_3 so that $(x(t),y(t))$ lies on the circle $x^2+y^2=4$?

2. Describe the motion of the particle if you set $a_2=a_4=1$ and let $0\le t\le 2\pi$. What is the starting point?

3. What choice of a_2 and a_4 will trace the circle twice in a counterclockwise direction starting at $(1,0)$? What choice will trace the circle three times in a counterclockwise direction?

4. Describe the motion if you set $a_2=1$, $a_4=-1$ and let $0\le t\le 2\pi$.

5. What choice of a_2 and a_4 will trace the circle three times in a clockwise direction starting at $(1,0)$? What choice will trace the circle five times in a clockwise direction?

GROUP WORK 3, SECTION 1.7
Lissajous Figures

The curves with parametric equations $x(t) = a_1 \cos a_2 t$, $y(t) = a_3 \sin a_4 t$ are called Lissajous Figures. In this exercise, we fix $a_1 = a_3 = 1$ and try to determine the effects of varying a_2 and a_4 on the shape of the figure.

1. Compare the graph of $x(t) = \cos 2t$, $y(t) = \sin 3t$ to the graph of $x(t) = \cos t$, $y(t) = \sin 1.5t$. Is there any difference in the shapes of the two figures? Is there any difference in how they are traced out? Now look at the graph of $x(t) = \cos 4t$, $y(t) = \sin 6t$. Can you make a generalization about figures where the ratio of a_2 to a_4 is some fixed value?

2. Fix $a_2 = 1$ and look at the graphs with $a_4 = 1$, 2, 3, and 4. What happens as a_4 increases? Can you predict how the figure will look if $a_2 = 1$ and $a_4 = 5$?

3. Now fix $a_4 = 1$ and look at the graphs with $a_2 = 1$, 2, 3, 4, and 5. Do you see the same behavior as in Part 2?

4. The figure with $a_4 = 1$ and $a_2 = 2$ should have a familiar geometric shape. What is this shape? Can you use a trigonometric identity for $x = \cos 2t$ to explain why the figure looks the way it does?

5. Now set $a_2 = 3$ and $a_4 = 2$. Starting at $t = 0$, what is the upper value b of t so that the figure is traced out exactly once between $t = 0$ and $t = b$? How many cycles does $x(t) = \cos 3t$ go through between $t = 0$ and $t = b$? How many cycles does $y(t) = \sin 2t$ go through? Can you see how these cycles are reflected in the shape of the Lissajous figure?

6. Set $a_2 = 7$ and $a_4 = 4$ and do the same analysis as in Part 5.

7. Repeat Parts 5 and 6 with the values of a_2 and a_4 reversed. Can you explain the difference in the shapes of the curves in this case?

Laboratory Project: Running Circles Around Circles

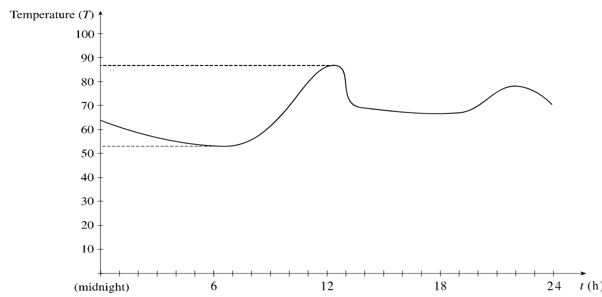
There is an interesting history behind epicycloids that can be used in introducing this project. After Copernicus showed that the sun didn't move around the Earth, astronomers believed that the planets moved in circular paths around the Sun. Gradually, mathematical analysis showed that this wasn't quite the path of the planets. So they posited that the "circular paths" were actually epicycles: small circles rolling around larger ones. (See Problems 5–7 in the project.) More accurate numerical data showed that this theory was also wrong. It was then believed that the paths were double-epicycles: circles rolling around circles rolling around circles. Finally, Kepler (using Brahe's data) showed that the paths were elliptical, and then Newton, using his newly developed calculus, derived laws to discover reasons why Kepler's discovery was true.

Problems 1–3 are relatively straightforward, with 4–7 being more of a challenge for the students. Any of the first three questions could easily be included in a regular assignment. We recommend assigning at least the first four problems if this is to be an extended project, with the remaining problems being used as extra credit if they are not mandatory. It would be helpful to have several large drawings of the various curves on paper ready for students who come to office hours with questions. (TEC can be helpful in answering Problems 2–4, 6, and 7.)

1 Sample Exam Questions

Problems marked with an asterisk (*) are particularly challenging and should be given careful consideration.

1. The graph below shows the temperature of a room during a summer day as a function of time, starting at midnight.



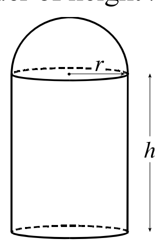
(a) Evaluate f (noon) and f (6 P.M.). State the range of f.

(b) Where is f increasing? Decreasing?

(c) Give a possible explanation for what happened at noon.

(d) Give a possible explanation why f attains its minimum value at 6 A.M.

2. A proposed new grain silo consists of a cylinder of height h and radius r, capped by a hemisphere.



Express its volume as a function of h and r.

*** 3.** The Slopps® trading card company has decided to put out its best line of trading cards ever: The "Famous Mathematicians" Series. Each pack of cards contains eight famous mathematicians, a mathematical puzzle, and a mathematics sticker. Naturally, you want a complete set, but you will have to buy a lot of cards because the really good ones (like the Galois, Sylvester, Hesse, Newton, and Leibniz cards) are very rare. Your local dealer will sell you an individual card, randomly selected, for 50 cents. Most people are interested in buying the packs of 8 for $\$2.80$. When you tell the dealer you want to buy a *lot* of them, he offers to sell you a box (containing 10 packs) for $\$25$, or a carton (containing 10 boxes) for $\$230$.
Let $c(x)$ be the (least) cost of buying x cards. Note that it is acceptable to buy more than x cards if it costs less than buying exactly x cards.

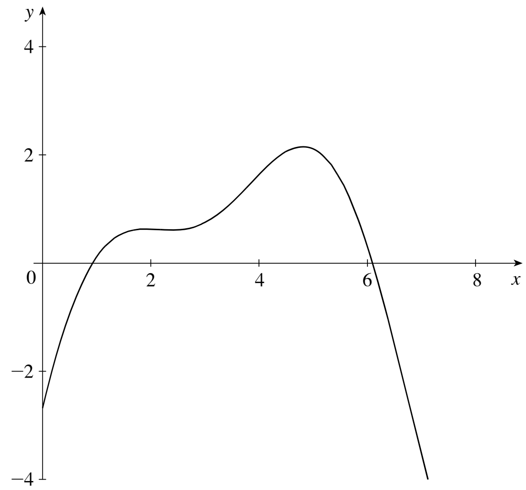
(a) Explain why the cheapest way to buy 6 cards is to buy a pack of 8. What is $c(6)$?

(b) Sketch a graph of $y = c(x)$ from $x = 0$ to $x = 24$.

(c) Find a formula for $c(x)$, valid for $x = 80$ to $x = 90$.

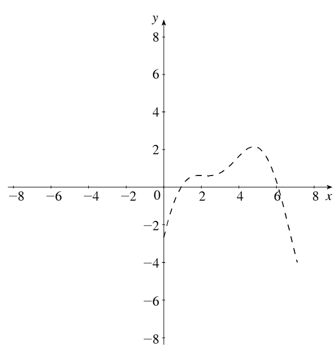
(d) If you wanted to buy 1005 cards, what is the least you would have to pay?

4. The following is a graph of $y = f(x)$.

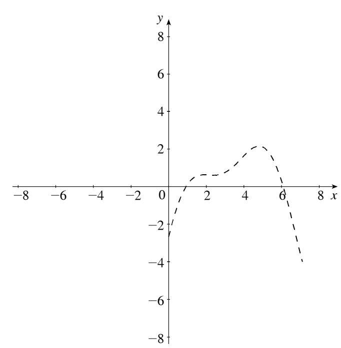


Draw and label graphs of

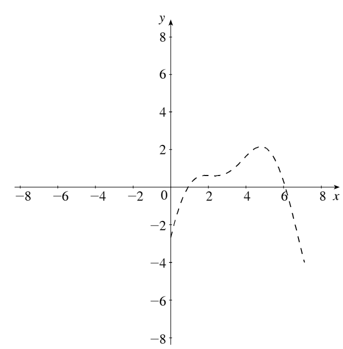
(a) $2f(x+2)$



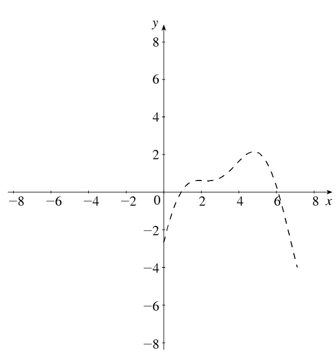
(b) $\frac{1}{2}f(-x)$



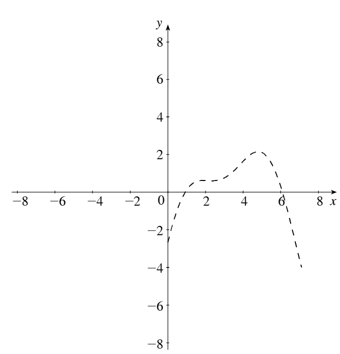
(c) $f(2x)$



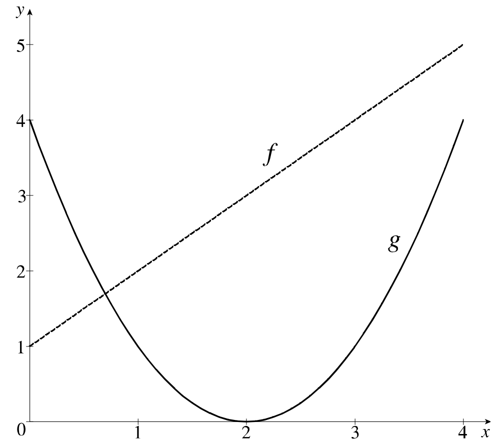
(d) $2f(2x-2)$



(e) $f(2x)-2$



5. Use the given graphs of f and g to evaluate each expression, or explain why it is undefined.



(a) $(f \circ g)(2)$

(b) $(g \circ f)(2)$

(c) $(f \circ f)(2)$

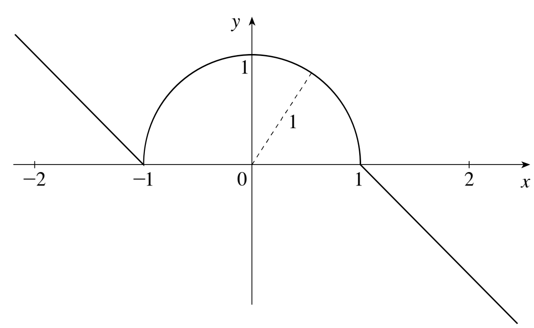
(d) $(g \circ g)(2)$

(e) $(f + g)(2)$

(f) $(f/g)(2)$

(g) $g^{-1}(2)$

6. Find a formula that describes the following function.



7. Let $f(x) = 2.912345x^2 + 3.131579x - 0.099999$.

(a) To approximate f, write a quadratic function $g(x)$ with *integer* coefficients that closely models $f(x)$ for $-10 < x < 10$.

(b) Compute $g(4)$ and $f(4)$.

(c) Compute the error in using $g(4)$ to approximate $f(4)$ as a percentage of the correct answer $f(4)$.

(d) For larger values of x (say $x = 10$ or $x = 20$), would $g(x)$ be an overestimate or an underestimate of $f(x)$? Justify your answer without computing specific values of f and g.

8. Let f be a one-to-one function whose *inverse* function is given by the formula

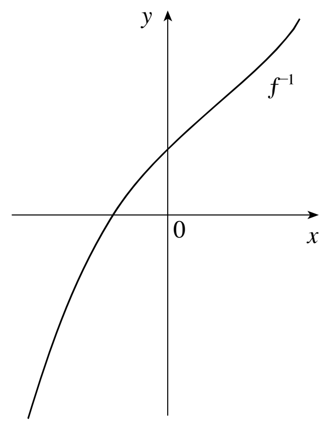
$$f^{-1}(x) = x^5 + 2x^3 + 3x + 1$$

(a) Compute $f^{-1}(1)$ and $f(1)$.

(b) Compute the value of x_0 such that $f(x_0) = 1$.

(c) Compute the value of y_0 such that $f^{-1}(y_0) = 1$.

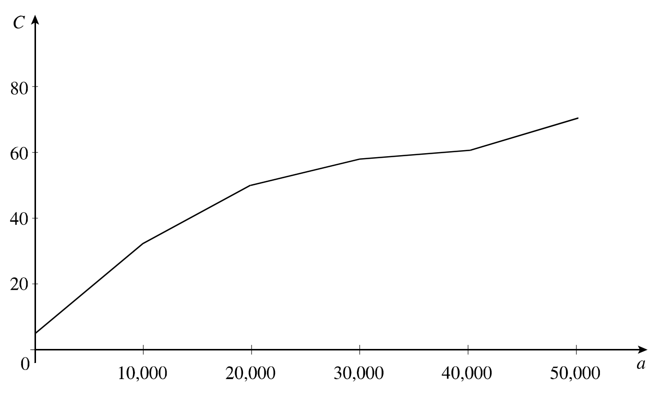
(d) Below is a graph of f^{-1}. Draw an approximate graph of f.



9. Find constants A, B, and k such that the equation $f(x) = A2^{kx} + B$ satisfies the following three conditions:

- $f(x)$ is always decreasing.
- $f(x)$ has a horizontal asymptote at $y = 1$, and
- $f(x)$ goes through the point $(0, 4)$.

10. A manufacturer hires a mathematician to come up with a function f that models the cost C of producing a compact discs, where C is in thousands of dollars. The graph of f is given below.



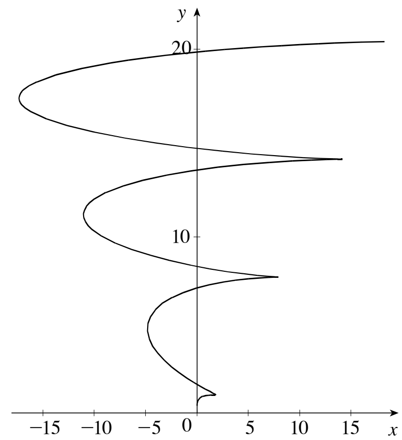
(a) What does $f(50{,}000) - f(49{,}999)$ represent?

(b) What does $f^{-1}(10)$ represent?

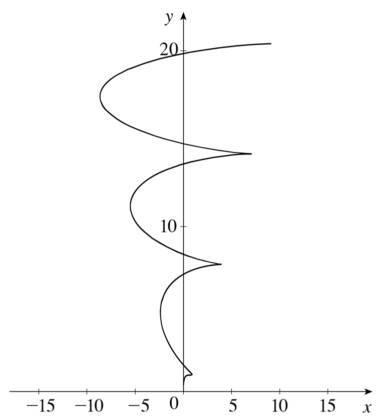
(c) For what value or range of values is the cost per disc the least?

(d) Give a possible explanation for the sudden increase in the curve's slope at the end.

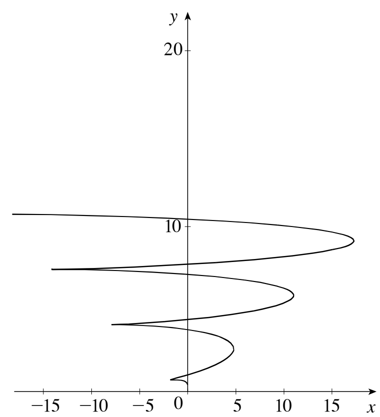
11. The curve below can be parametrized by $x(t) = t\sin t$, $y(t) = t + \cos t$.



(a) Give a parametrization for the curve below.



(b) Give a parametrization for the curve below.



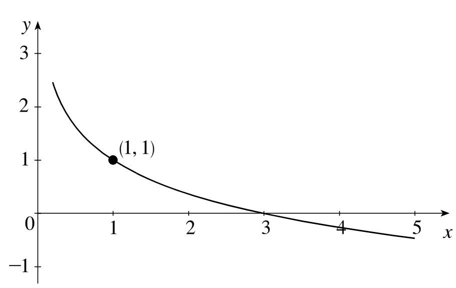
12. Draw a graph of $f(x) = \ln\ln x$

(a) Over the range $[2, 10]$.

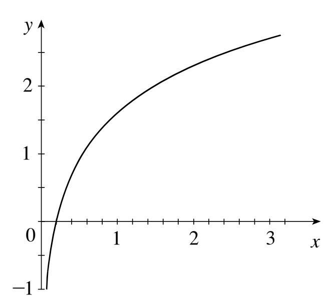
(b) Over the range $[2, 100]$.

(c) What is $\lim_{x\to\infty} \ln\ln x$?

13. (a) Below is a graph of $f(x) = A\ln x + B$. What are A and B?



(b) Below is a graph of $f(x) = \ln kx$. What is k?



1 Sample Exam Solutions

1. Approximate answers are acceptable for this problem.

(a) $f\,(\text{noon}) = 87°$, $f\,(6 \text{ P.M.}) = 67°$, range of f is $[53, 87]$.

(b) f is increasing on $(6, 12)$ and $(20, 22)$; f is decreasing on $(0, 6)$, $(12, 20)$ and $(22, 24)$.

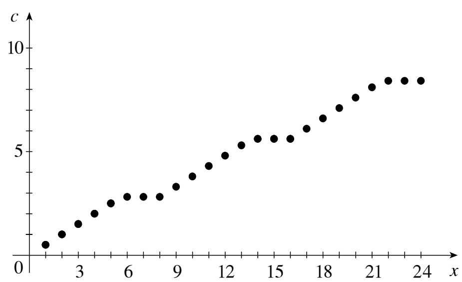
(c) Possible explanations for the drop in temperature at noon are a sudden thundershower, or an air conditioner being turned on.

(d) A possible explanation for f attaining its minimum value at 6 A.M. is that this is just before sunrise.

2. The total volume is the volume of a cylinder of height and radius r plus the volume of a hemisphere of radius r, that is, $V = \pi r^2 h + \frac{2}{3}\pi r^3$.

3. (a) If we buy 8 cards for $\$2.80$, then this costs less than buying 6 individual cards at $\$0.50$ apiece. Hence, $C\,(6) = \$2.80$.

(b)

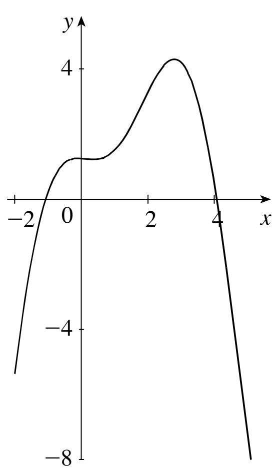


(c)

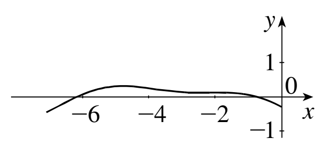
$$c\,(x) = \begin{cases} 25 + 0.5\,(x - 80) & \text{if } 80 \le x \le 85 \\ 27.8 & \text{if } 86 \le x \le 88 \\ 27.8 + 0.5\,(x - 88) & \text{if } 89 \le x \le 90 \end{cases}$$

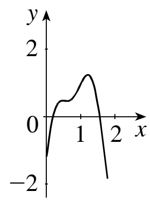
(d) To buy 1005 cards, the best deal is to buy one carton (800 cards), two boxes (160 cards), five packs (40 cards) and five individual cards. The total cost is $230 + 2\,(25) + 5\,(2.80) + 5\,(0.50) = \$296.50$.

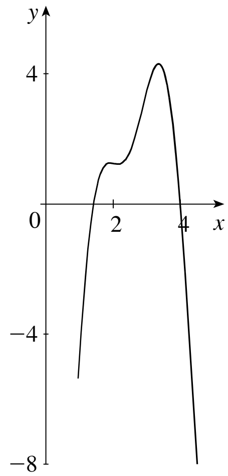
4. (a)

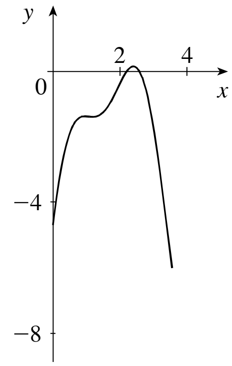


(b)



(c) 

(d) 

(e) 

5. (a) $(f \circ g)(2) = f(0) = 1$

(b) $(g \circ f)(2) = g(3) = 1$

(c) $(f \circ f)(2) = f(3) = 4$

(d) $(g \circ g)(2) = g(0) = 4$

(e) $(f + g)(2) = f(2) + g(2) = 3 + 0 = 3$

(f) $\left(\dfrac{f}{g}\right)(2)$ is undefined because $g(2) = 0$.

(g) $g^{-1}(2)$ is undefined because $g(x)$ takes on the value 2 twice, for $x = 0.6$ and $x = 3.4$.

6. $f(x) = \begin{cases} -x - 1 & \text{if } x < -1 \\ \sqrt{1 - x^2} & \text{if } -1 \le x \le 1 \\ -x + 1 & \text{if } x > 1 \end{cases}$

7. (a) $g(x) = 3x^2 + 3x$

(b) $g(4) = 60$, $f(4) = 59.023837$

(c) The percentage error in using $g(4)$ as an approximation for $f(4)$ is $100\left|\dfrac{f(4) - g(4)}{f(4)}\right| = 1.65\%$.

(d) For larger values of x, $g(x)$ is an overestimate of $f(x)$ because the coefficient of the dominant term (x^2) is larger.

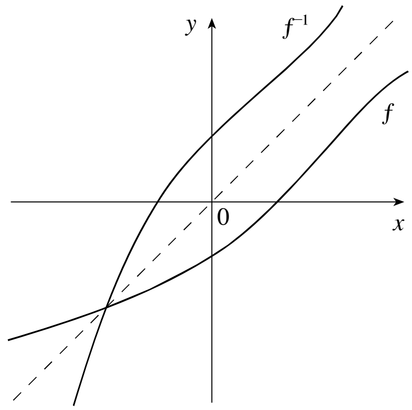
8. $f^{-1}(x) = x^5 + 2x^3 + 3x + 1$

(a) $f^{-1}(1) = 7$, $f(1) = 0$

(b) The value x_0 such that $f(x_0) = 1$ is $f^{-1}(1) = 7$.

(c) The value y_0 such that $f^{-1}(y_0) = 1$ is $f(1) = 0$.

(d) The graph of $f(x)$ is the graph of $f^{-1}(x)$ reflected about the line $y = x$.



9. Let $f(x) = 3(2)^{-x} + 1$. Then $f(x)$ is always decreasing, has a horizontal asymptote at $y = 1$, and $f(0) = 4$.

10. (a) $f(50{,}000) - f(49{,}999)$ represents the cost of producing the 50,000th disc.

(b) $f^{-1}(10)$ represents the number of discs that can be made for \$10,000.

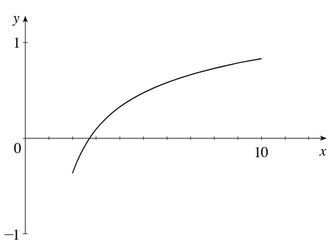
(c) The cost per disc is cheapest for $30{,}000 < a < 40{,}000$. This is where the slope of f is the smallest.

(d) One possible explanation for the sudden increase in the curve's slope is scarcity of materials.

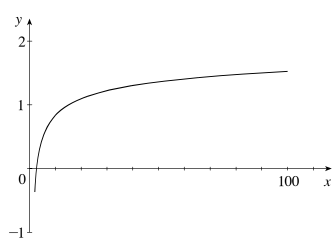
11. (a) $x_1(t) = \frac{1}{2}x(t) = \frac{1}{2}(t \sin t)$, $y_1(t) = y(t) = t + \cos t$

(b) $x_2(t) = -x(t) = -t \sin t$, $y_2(t) = \frac{1}{2}y(t) = \frac{1}{2}(t + \cos t)$

12. (a)



(b)



(c) $\lim\limits_{x \to \infty} \ln \ln x = \infty$

13. (a) Use $f(1) = 1$ and $f(3) = 0$ to get $A = -\dfrac{1}{\ln 3}$ and $B = 1$.

(b) Use $f(0.2) = 0$ to get $K = 5$.