Solutions to Problems, Section 1.1

The problems in this section may be harder than typical problems found in the rest of this book.

1. Show that $\frac{6}{7} + \sqrt{2}$ is an irrational number.

   SOLUTION Suppose $\frac{6}{7} + \sqrt{2}$ is a rational number. Because

   $$\sqrt{2} = (\tfrac{6}{7} + \sqrt{2}) - \tfrac{6}{7},$$

   this implies that $\sqrt{2}$ is the difference of two rational numbers, which implies that $\sqrt{2}$ is a rational number, which is not true. Thus our assumption that $\frac{6}{7} + \sqrt{2}$ is a rational number must be false. In other words, $\frac{6}{7} + \sqrt{2}$ is an irrational number.

2. Show that $5 - \sqrt{2}$ is an irrational number.

   SOLUTION Suppose $5 - \sqrt{2}$ is a rational number. Because

   $$\sqrt{2} = 5 - (5 - \sqrt{2}),$$

   this implies that $\sqrt{2}$ is the difference of two rational numbers, which implies that $\sqrt{2}$ is a rational number, which is not true. Thus our assumption that $5 - \sqrt{2}$ is a rational number must be false. In other words, $5 - \sqrt{2}$ is an irrational number.

3. Show that $3\sqrt{2}$ is an irrational number.

   SOLUTION Suppose $3\sqrt{2}$ is a rational number. Because

   $$\sqrt{2} = \frac{3\sqrt{2}}{3},$$

   this implies that $\sqrt{2}$ is the quotient of two rational numbers, which implies that $\sqrt{2}$ is a rational number, which is not true. Thus our assumption that $3\sqrt{2}$ is a rational number must be false. In other words, $3\sqrt{2}$ is an irrational number.

4. Show that $\frac{3\sqrt{2}}{5}$ is an irrational number.

   SOLUTION Suppose $\frac{3\sqrt{2}}{5}$ is a rational number. Because

   $$\sqrt{2} = \frac{3\sqrt{2}}{5} \cdot \frac{5}{3},$$

   this implies that $\sqrt{2}$ is the product of two rational numbers, which implies that $\sqrt{2}$ is a rational number, which is not true. Thus our assumption that $\frac{3\sqrt{2}}{5}$ is a rational number must be false. In other words, $\frac{3\sqrt{2}}{5}$ is an irrational number.

5. Show that $4 + 9\sqrt{2}$ is an irrational number.

   SOLUTION Suppose $4 + 9\sqrt{2}$ is a rational number. Because

   $$9\sqrt{2} = (4 + 9\sqrt{2}) - 4,$$

this implies that $9\sqrt{2}$ is the difference of two rational numbers, which implies that $9\sqrt{2}$ is a rational number. Because

$$\sqrt{2} = \frac{9\sqrt{2}}{9},$$

this implies that $\sqrt{2}$ is the quotient of two rational numbers, which implies that $\sqrt{2}$ is a rational number, which is not true. Thus our assumption that $4 + 9\sqrt{2}$ is a rational number must be false. In other words, $4 + 9\sqrt{2}$ is an irrational number.

6. Explain why the sum of a rational number and an irrational number is an irrational number.

   SOLUTION We have already seen the pattern for this solution in Problems 1 and 2. We can repeat that pattern, using arbitrary numbers instead of specific numbers.

   Suppose r is a rational number and x is an irrational number. We need to show that $r + x$ is an irrational number.

   Suppose $r + x$ is a rational number. Because

$$x = (r + x) - r,$$

   this implies that x is the difference of two rational numbers, which implies that x is a rational number, which is not true. Thus our assumption that $r + x$ is a rational number must be false. In other words, $r + x$ is an irrational number.

7. Explain why the product of a nonzero rational number and an irrational number is an irrational number.

   SOLUTION We have already seen the pattern for this solution in Problems 3 and 4. We can repeat that pattern, using arbitrary numbers instead of specific numbers.

   Suppose r is a nonzero rational number and x is an irrational number. We need to show that rx is an irrational number.

   Suppose rx is a rational number. Because

$$x = \frac{rx}{r}$$

   this implies that x is the quotient of two rational numbers, which implies that x is a rational number, which is not true. Thus our assumption that rx is a rational number must be false. In other words, rx is an irrational number.

   Note that the hypothesis that r is nonzero is needed because otherwise we would be dividing by 0 in the equation above.

8. Suppose t is an irrational number. Explain why $\frac{1}{t}$ is also an irrational number.

   SOLUTION Suppose $\frac{1}{t}$ is a rational number. Then there exist integers m and n, with $n \neq 0$, such that

$$\frac{1}{t} = \frac{m}{n}.$$

   Note that $m \neq 0$, because $\frac{1}{t}$ cannot equal 0.

The equation above implies that

$$t = \frac{n}{m},$$

which implies that t is a rational number, which is not true. Thus our assumption that $\frac{1}{t}$ is a rational number must be false. In other words, $\frac{1}{t}$ is an irrational number.

9. Give an example of two irrational numbers whose sum is an irrational number.

SOLUTION Problem 7 implies that $2\sqrt{2}$ and $3\sqrt{2}$ are irrational numbers. Because

$$\sqrt{2} + 2\sqrt{2} = 3\sqrt{2},$$

we have an example of two irrational numbers whose sum is an irrational number.

10. Give an example of two irrational numbers whose sum is a rational number.

SOLUTION Note that

$$\sqrt{2} + (5 - \sqrt{2}) = 5.$$

Thus we have two irrational numbers ($5 - \sqrt{2}$ is irrational by Problem 2) whose sum equals a rational number.

11. Give an example of three irrational numbers whose sum is a rational number.

SOLUTION Here is one example among many possibilities:

$$(5 - \sqrt{2}) + (4 - \sqrt{2}) + 2\sqrt{2} = 9.$$

12. Give an example of two irrational numbers whose product is an irrational number.

SOLUTION Here is one example among many possibilities:

$$(5 - \sqrt{2})\sqrt{2} = 5\sqrt{2} - 2.$$

13. Give an example of two irrational numbers whose product is a rational number.

SOLUTION Here is one example among many possibilities:

$$(3\sqrt{2})\sqrt{2} = 3 \cdot \sqrt{2}^2 = 3 \cdot 2 = 6.$$

Solutions to Exercises, Section 1.2

For Exercises 1–4, determine how many different values can arise by inserting one pair of parentheses into the given expression.

1. $19 - 12 - 8 - 2$

SOLUTION Here are the possibilities:

$$19(-12 - 8 - 2) = -418 \qquad 19 - (12 - 8) - 2 = 13$$
$$19(-12 - 8) - 2 = -382 \qquad 19 - (12 - 8 - 2) = 17$$
$$19(-12) - 8 - 2 = -238 \qquad 19 - 12 - 8(-2) = 23$$
$$(19 - 12) - 8 - 2 = -3 \qquad 19 - 12(-8) - 2 = 113$$
$$19 - 12 - (8 - 2) = 1 \qquad 19 - 12(-8 - 2) = 139$$

Other possible ways to insert one pair of parentheses lead to values already included in the list above. For example,

$$(19 - 12 - 8) - 2 = -3.$$

Thus ten values are possible; they are -418, -382, -238, -3, 1, 13, 17, 23, 113, and 139.

2. $3 - 7 - 9 - 5$

SOLUTION Here are the possibilities:

$$3(-7 - 9 - 5) = -63$$
$$3(-7 - 9) - 5 = -53$$
$$3(-7) - 9 - 5 = -35$$
$$(3 - 7) - 9 - 5 = -18$$
$$3 - 7 - (9 - 5) = -8$$
$$3 - (7 - 9) - 5 = 0$$
$$3 - (7 - 9 - 5) = 10$$
$$3 - 7 - 9(-5) = 41$$
$$3 - 7(-9) - 5 = 61$$
$$3 - 7(-9 - 5) = 101$$

Other possible ways to insert one pair of parentheses lead to values already included in the list above. For example,

$$(3 - 7 - 9) - 5 = -18.$$

Thus ten values are possible; they are -63, -53, -35, -18, -8, 0, 10, 41, 61, and 101.

3. $6 + 3 \cdot 4 + 5 \cdot 2$

SOLUTION Here are the possibilities:

$$(6+3\cdot 4+5\cdot 2)=28$$
$$6+(3\cdot 4+5)\cdot 2=40$$
$$(6+3)\cdot 4+5\cdot 2=46$$
$$6+3\cdot(4+5\cdot 2)=48$$
$$6+3\cdot(4+5)\cdot 2=60$$

Other possible ways to insert one pair of parentheses lead to values already included in the list above. For example,

$$(6+3\cdot 4+5)\cdot 2=46.$$

Thus five values are possible; they are 28, 40, 46, 48, and 60.

4. $5\cdot 3\cdot 2+6\cdot 4$

SOLUTION Here are the possibilities:

$$(5\cdot 3\cdot 2+6\cdot 4)=54$$
$$(5\cdot 3\cdot 2+6)\cdot 4=144$$
$$5\cdot(3\cdot 2+6\cdot 4)=150$$
$$5\cdot(3\cdot 2+6)\cdot 4=240$$
$$5\cdot 3\cdot(2+6\cdot 4)=390$$
$$5\cdot 3\cdot(2+6)\cdot 4=480$$

Other possible ways to insert one pair of parentheses lead to values already included in the list above. For example,

$$(5\cdot 3)\cdot 2+6\cdot 4=54.$$

Thus six values are possible; they are 54, 144, 150, 240, 390, and 480.

For Exercises 5–22, expand the given expression.

5. $(x-y)(z+w-t)$

SOLUTION

$$\begin{aligned}(x-y)(z+w-t)&\\ &=x(z+w-t)-y(z+w-t)\\ &=xz+xw-xt-yz-yw+yt\end{aligned}$$

6. $(x + y - r)(z + w - t)$

SOLUTION

$$\begin{aligned}&(x + y - r)(z + w - t)\\&\quad = x(z + w - t) + y(z + w - t) - r(z + w - t)\\&\quad = xz + xw - xt + yz + yw - yt - rz - rw + rt\end{aligned}$$

7. $(2x + 3)^2$

SOLUTION

$$\begin{aligned}(2x + 3)^2 &= (2x)^2 + 2 \cdot (2x) \cdot 3 + 3^2\\&= 4x^2 + 12x + 9\end{aligned}$$

8. $(3b + 5)^2$

SOLUTION

$$\begin{aligned}(3b + 5)^2 &= (3b)^2 + 2 \cdot (3b) \cdot 5 + 5^2\\&= 9b^2 + 30b + 25\end{aligned}$$

9. $(2c - 7)^2$

SOLUTION

$$\begin{aligned}(2c - 7)^2 &= (2c)^2 - 2 \cdot (2c) \cdot 7 + 7^2\\&= 4c^2 - 28c + 49\end{aligned}$$

10. $(4a - 5)^2$

SOLUTION

$$\begin{aligned}(4a - 5)^2 &= (4a)^2 - 2 \cdot (4a) \cdot 5 + 5^2\\&= 16a^2 - 40a + 25\end{aligned}$$

11. $(x + y + z)^2$

SOLUTION

$$\begin{aligned}
&(x+y+z)^2\\
&\quad= (x+y+z)(x+y+z)\\
&\quad= x(x+y+z)+y(x+y+z)+z(x+y+z)\\
&\quad= x^2+xy+xz+yx+y^2+yz\\
&\qquad\quad +zx+zy+z^2\\
&\quad= x^2+y^2+z^2+2xy+2xz+2yz
\end{aligned}$$

12. $(x-5y-3z)^2$

SOLUTION

$$\begin{aligned}
&(x-5y-3z)^2\\
&\quad= (x-5y-3z)(x-5y-3z)\\
&\quad= x(x-5y-3z)-5y(x-5y-3z)\\
&\qquad -3z(x-5y-3z)\\
&\quad= x^2-5xy-3xz-5yx+25y^2+15yz\\
&\qquad -3zx+15zy+9z^2\\
&\quad= x^2+25y^2+9z^2-10xy-6xz+30yz
\end{aligned}$$

13. $(x+1)(x-2)(x+3)$

SOLUTION

$$\begin{aligned}
&(x+1)(x-2)(x+3)\\
&\qquad\qquad= \big((x+1)(x-2)\big)(x+3)\\
&\qquad\qquad= (x^2-2x+x-2)(x+3)\\
&\qquad\qquad= (x^2-x-2)(x+3)\\
&\qquad\qquad= x^3+3x^2-x^2-3x-2x-6\\
&\qquad\qquad= x^3+2x^2-5x-6
\end{aligned}$$

14. $(y-2)(y-3)(y+5)$

SOLUTION

$$\begin{aligned}(y-2)(y-3)(y+5) &= \big((y-2)(y-3)\big)(y+5)\\ &= (y^2-3y-2y+6)(y+5)\\ &= (y^2-5y+6)(y+5)\\ &= y^3+5y^2-5y^2-25y+6y+30\\ &= y^3-19y+30\end{aligned}$$

15. $(a+2)(a-2)(a^2+4)$

SOLUTION

$$\begin{aligned}(a+2)(a-2)(a^2+4) &= \big((a+2)(a-2)\big)(a^2+4)\\ &= (a^2-4)(a^2+4)\\ &= a^4-16\end{aligned}$$

16. $(b-3)(b+3)(b^2+9)$

SOLUTION

$$\begin{aligned}(b-3)(b+3)(b^2+9) &= \big((b-3)(b+3)\big)(b^2+9)\\ &= (b^2-9)(b^2+9)\\ &= b^4-81\end{aligned}$$

17. $xy(x+y)(\frac{1}{x}-\frac{1}{y})$

SOLUTION

$$\begin{aligned}xy(x+y)\Big(\frac{1}{x}-\frac{1}{y}\Big) &= xy(x+y)\Big(\frac{y}{xy}-\frac{x}{xy}\Big)\\ &= xy(x+y)\Big(\frac{y-x}{xy}\Big)\\ &= (x+y)(y-x)\\ &= y^2-x^2\end{aligned}$$

18. $a^2z(z-a)(\frac{1}{z}+\frac{1}{a})$

SOLUTION

$$\begin{aligned} a^2z(z-a)\left(\frac{1}{z}+\frac{1}{a}\right) &= a^2z(z-a)\left(\frac{a}{az}+\frac{z}{az}\right) \\ &= a^2z(z-a)\left(\frac{a+z}{az}\right) \\ &= a(z-a)(a+z) \\ &= a(z^2-a^2) \\ &= az^2-a^3 \end{aligned}$$

19. $(t-2)(t^2+2t+4)$

SOLUTION

$$\begin{aligned} (t-2)(t^2+2t+4) &= t(t^2+2t+4)-2(t^2+2t+4) \\ &= t^3+2t^2+4t-2t^2-4t-8 \\ &= t^3-8 \end{aligned}$$

20. $(m-2)(m^4+2m^3+4m^2+8m+16)$

SOLUTION

$$\begin{aligned} &(m-2)(m^4+2m^3+4m^2+8m+16) \\ &\quad = m(m^4+2m^3+4m^2+8m+16)-2(m^4+2m^3+4m^2+8m+16) \\ &\quad = m^5+2m^4+4m^3+8m^2+16m-2m^4-4m^3-8m^2-16m-32 \\ &\quad = m^5-32 \end{aligned}$$

21. $(n+3)(n^2-3n+9)$

SOLUTION

$$\begin{aligned} &(n+3)(n^2-3n+9) \\ &\quad = n(n^2-3n+9)+3(n^2-3n+9) \\ &\quad = n^3-3n^2+9n+3n^2-9n+27 \\ &\quad = n^3+27 \end{aligned}$$

22. $(y+2)(y^4-2y^3+4y^2-8y+16)$

SOLUTION

$$\begin{aligned}
&(y+2)(y^4-2y^3+4y^2-8y+16)\\
&\quad = y(y^4-2y^3+4y^2-8y+16)+2(y^4-2y^3+4y^2-8y+16)\\
&\quad = y^5-2y^4+4y^3-8y^2+16y+2y^4-4y^3+8y^2-16y+32\\
&\quad = y^5+32
\end{aligned}$$

For Exercises 23–50, simplify the given expression as much as possible.

23. $4(2m+3n)+7m$

SOLUTION

$$\begin{aligned}
4(2m+3n)+7m &= 8m+12n+7m\\
&= 15m+12n
\end{aligned}$$

24. $3\big(2m+4(n+5p)\big)+6n$

SOLUTION

$$\begin{aligned}
3\big(2m+4(n+5p)\big)+6n &= 6m+12(n+5p)+6n\\
&= 6m+12n+60p+6n\\
&= 6m+18n+60p
\end{aligned}$$

25. $\dfrac{3}{4}+\dfrac{6}{7}$

SOLUTION $\dfrac{3}{4}+\dfrac{6}{7}=\dfrac{3}{4}\cdot\dfrac{7}{7}+\dfrac{6}{7}\cdot\dfrac{4}{4}=\dfrac{21}{28}+\dfrac{24}{28}=\dfrac{45}{28}$

26. $\dfrac{2}{5}+\dfrac{7}{8}$

SOLUTION $\dfrac{2}{5}+\dfrac{7}{8}=\dfrac{2}{5}\cdot\dfrac{8}{8}+\dfrac{7}{8}\cdot\dfrac{5}{5}=\dfrac{16}{40}+\dfrac{35}{40}=\dfrac{51}{40}$

27. $\dfrac{3}{4}\cdot\dfrac{14}{39}$

SOLUTION $\dfrac{3}{4}\cdot\dfrac{14}{39}=\dfrac{3\cdot 14}{4\cdot 39}=\dfrac{7}{2\cdot 13}=\dfrac{7}{26}$

28. $\dfrac{2}{3} \cdot \dfrac{15}{22}$

SOLUTION $\dfrac{2}{3} \cdot \dfrac{15}{22} = \dfrac{2 \cdot 15}{3 \cdot 22} = \dfrac{5}{11}$

29. $\dfrac{\frac{5}{7}}{\frac{2}{3}}$

SOLUTION $\dfrac{\frac{5}{7}}{\frac{2}{3}} = \dfrac{5}{7} \cdot \dfrac{3}{2} = \dfrac{5 \cdot 3}{7 \cdot 2} = \dfrac{15}{14}$

30. $\dfrac{\frac{6}{5}}{\frac{7}{4}}$

SOLUTION $\dfrac{\frac{6}{5}}{\frac{7}{4}} = \dfrac{6}{5} \cdot \dfrac{4}{7} = \dfrac{6 \cdot 4}{5 \cdot 7} = \dfrac{24}{35}$

31. $\dfrac{m+1}{2} + \dfrac{3}{n}$

SOLUTION

$$\begin{aligned}
\frac{m+1}{2} + \frac{3}{n} &= \frac{m+1}{2} \cdot \frac{n}{n} + \frac{3}{n} \cdot \frac{2}{2} \\
&= \frac{(m+1)n + 3 \cdot 2}{2n} \\
&= \frac{mn + n + 6}{2n}
\end{aligned}$$

32. $\dfrac{m}{3} + \dfrac{5}{n-2}$

SOLUTION

$$\begin{aligned}
\frac{m}{3} + \frac{5}{n-2} &= \frac{m}{3} \cdot \frac{n-2}{n-2} + \frac{5}{n-2} \cdot \frac{3}{3} \\
&= \frac{m(n-2) + 15}{3(n-2)} \\
&= \frac{mn - 2m + 15}{3n - 6}
\end{aligned}$$

33. $\dfrac{2}{3} \cdot \dfrac{4}{5} + \dfrac{3}{4} \cdot 2$

SOLUTION

$$
\begin{aligned}
\frac{2}{3} \cdot \frac{4}{5} + \frac{3}{4} \cdot 2 &= \frac{8}{15} + \frac{3}{2} \\
&= \frac{8}{15} \cdot \frac{2}{2} + \frac{3}{2} \cdot \frac{15}{15} \\
&= \frac{16 + 45}{30} \\
&= \frac{61}{30}
\end{aligned}
$$

34. $\frac{3}{5} \cdot \frac{2}{7} + \frac{5}{4} \cdot 2$

SOLUTION

$$
\begin{aligned}
\frac{3}{5} \cdot \frac{2}{7} + \frac{5}{4} \cdot 2 &= \frac{6}{35} + \frac{5}{2} \\
&= \frac{6}{35} \cdot \frac{2}{2} + \frac{5}{2} \cdot \frac{35}{35} \\
&= \frac{12 + 175}{70} \\
&= \frac{187}{70}
\end{aligned}
$$

35. $\frac{2}{5} \cdot \frac{m+3}{7} + \frac{1}{2}$

SOLUTION

$$
\begin{aligned}
\frac{2}{5} \cdot \frac{m+3}{7} + \frac{1}{2} &= \frac{2m+6}{35} + \frac{1}{2} \\
&= \frac{2m+6}{35} \cdot \frac{2}{2} + \frac{1}{2} \cdot \frac{35}{35} \\
&= \frac{4m + 12 + 35}{70} \\
&= \frac{4m + 47}{70}
\end{aligned}
$$

36. $\frac{3}{4} \cdot \frac{n-2}{5} + \frac{7}{3}$

SOLUTION

$$\begin{aligned}\frac{3}{4}\cdot\frac{n-2}{5}+\frac{7}{3} &= \frac{3n-6}{20}+\frac{7}{3}\\ &= \frac{3n-6}{20}\cdot\frac{3}{3}+\frac{7}{3}\cdot\frac{20}{20}\\ &= \frac{9n-18+140}{60}\\ &= \frac{9n+122}{60}\end{aligned}$$

37. $\dfrac{2}{x+3}+\dfrac{y-4}{5}$

SOLUTION

$$\begin{aligned}\frac{2}{x+3}+\frac{y-4}{5} &= \frac{2}{x+3}\cdot\frac{5}{5}+\frac{y-4}{5}\cdot\frac{x+3}{x+3}\\ &= \frac{2\cdot 5+(y-4)(x+3)}{5(x+3)}\\ &= \frac{10+yx+3y-4x-12}{5(x+3)}\\ &= \frac{xy-4x+3y-2}{5(x+3)}\end{aligned}$$

38. $\dfrac{x-3}{4}-\dfrac{5}{y+2}$

SOLUTION

$$\begin{aligned}\frac{x-3}{4}-\frac{5}{y+2} &= \frac{x-3}{4}\cdot\frac{y+2}{y+2}-\frac{5}{y+2}\cdot\frac{4}{4}\\ &= \frac{(x-3)(y+2)-5\cdot 4}{4(y+2)}\\ &= \frac{xy+2x-3y-6-20}{4(y+2}\\ &= \frac{xy+2x-3y-26}{4(y+2)}\end{aligned}$$

39. $\dfrac{4t+1}{t^2}+\dfrac{3}{t}$

SOLUTION

$$\begin{aligned}\frac{4t+1}{t^2}+\frac{3}{t} &= \frac{4t+1}{t^2}+\frac{3}{t}\cdot\frac{t}{t}\\ &= \frac{4t+1}{t^2}+\frac{3t}{t^2}\\ &= \frac{7t+1}{t^2}\end{aligned}$$

40. $\dfrac{5}{u^2}+\dfrac{1-2u}{u^3}$

SOLUTION

$$\begin{aligned}\frac{5}{u^2}+\frac{1-2u}{u^3} &= \frac{5}{u^2}\cdot\frac{u}{u}+\frac{1-2u}{u^3}\\ &= \frac{5u}{u^3}+\frac{1-2u}{u^3}\\ &= \frac{3u+1}{u^3}\end{aligned}$$

41. $\dfrac{3}{v(v-2)}+\dfrac{v+1}{v^3}$

SOLUTION

$$\begin{aligned}\frac{3}{v(v-2)}+\frac{v+1}{v^3} &= \frac{v^2}{v^2}\cdot\frac{3}{v(v-2)}+\frac{v+1}{v^3}\cdot\frac{v-2}{v-2}\\ &= \frac{3v^2}{v^3(v-2)}+\frac{v^2-v-2}{v^3(v-2)}\\ &= \frac{4v^2-v-2}{v^3(v-2)}\end{aligned}$$

42. $\dfrac{w-1}{w^3}-\dfrac{2}{w(w-3)}$

SOLUTION

$$\begin{aligned}\frac{w-1}{w^3}-\frac{2}{w(w-3)}&=\frac{w-1}{w^3}\cdot\frac{w-3}{w-3}-\frac{w^2}{w^2}\cdot\frac{2}{w(w-3)}\\&=\frac{(w-1)(w-3)}{w^3(w-3)}-\frac{2w^2}{w^3(w-3)}\\&=\frac{w^2-4w+3}{w^3(w-3)}-\frac{2w^2}{w^3(w-3)}\\&=\frac{-w^2-4w+3}{w^3(w-3)}\end{aligned}$$

43. $\dfrac{1}{x-y}\left(\dfrac{x}{y}-\dfrac{y}{x}\right)$

SOLUTION

$$\begin{aligned}\frac{1}{x-y}\left(\frac{x}{y}-\frac{y}{x}\right)&=\frac{1}{x-y}\left(\frac{x}{y}\cdot\frac{x}{x}-\frac{y}{x}\cdot\frac{y}{y}\right)\\&=\frac{1}{x-y}\left(\frac{x^2-y^2}{xy}\right)\\&=\frac{1}{x-y}\left(\frac{(x+y)(x-y)}{xy}\right)\\&=\frac{x+y}{xy}\end{aligned}$$

44. $\dfrac{1}{y}\left(\dfrac{1}{x-y}-\dfrac{1}{x+y}\right)$

SOLUTION

$$\begin{aligned}&\frac{1}{y}\left(\frac{1}{x-y}-\frac{1}{x+y}\right)\\&\quad=\frac{1}{y}\left(\frac{1}{x-y}\cdot\frac{x+y}{x+y}-\frac{1}{x+y}\cdot\frac{x-y}{x-y}\right)\\&\quad=\frac{1}{y}\left(\frac{x+y-(x-y)}{x^2-y^2}\right)\\&\quad=\frac{1}{y}\left(\frac{2y}{x^2-y^2}\right)\\&\quad=\frac{2}{x^2-y^2}\end{aligned}$$

45. $\dfrac{(x+a)^2-x^2}{a}$

SOLUTION

$$\frac{(x+a)^2 - x^2}{a} = \frac{x^2 + 2xa + a^2 - x^2}{a}$$
$$= \frac{2xa + a^2}{a}$$
$$= 2x + a$$

46. $\dfrac{\frac{1}{x+a} - \frac{1}{x}}{a}$

SOLUTION

$$\frac{\frac{1}{x+a} - \frac{1}{x}}{a} = \frac{\frac{1}{x+a} \cdot \frac{x}{x} - \frac{1}{x} \cdot \frac{x+a}{x+a}}{a}$$
$$= \frac{\frac{x-(x+a)}{x(x+a)}}{a}$$
$$= \frac{\frac{-a}{x(x+a)}}{a}$$
$$= \frac{-1}{x(x+a)}$$

47. $\dfrac{\frac{x-2}{y}}{\frac{z}{x+2}}$

SOLUTION

$$\frac{\frac{x-2}{y}}{\frac{z}{x+2}} = \frac{x-2}{y} \cdot \frac{x+2}{z}$$
$$= \frac{x^2 - 4}{yz}$$

48. $\dfrac{\frac{x-4}{y+3}}{\frac{y-3}{x+4}}$

SOLUTION

$$\frac{\frac{x-4}{y+3}}{\frac{y-3}{x+4}} = \frac{x-4}{y+3} \cdot \frac{x+4}{y-3}$$
$$= \frac{x^2 - 16}{y^2 - 9}$$

49. $\dfrac{\frac{a-t}{b-c}}{\frac{b+c}{a+t}}$

SOLUTION

$$\begin{aligned}\frac{\frac{a-t}{b-c}}{\frac{b+c}{a+t}} &= \frac{a-t}{b-c}\cdot\frac{a+t}{b+c}\\ &= \frac{a^2-t^2}{b^2-c^2}\end{aligned}$$

50. $\dfrac{\frac{r+m}{u-n}}{\frac{n+u}{m-r}}$

SOLUTION

$$\begin{aligned}\frac{\frac{r+m}{u-n}}{\frac{n+u}{m-r}} &= \frac{r+m}{u-n}\cdot\frac{m-r}{n+u}\\ &= \frac{m^2-r^2}{u^2-n^2}\end{aligned}$$

Solutions to Problems, Section 1.2

51. Show that $(a+1)^2 = a^2 + 1$ if and only if $a = 0$.

SOLUTION Note that

$$(a+1)^2 = a^2 + 2a + 1.$$

Thus $(a+1)^2 = a^2 + 1$ if and only if $2a = 0$, which happens if and only if $a = 0$.

52. Explain why $(a+b)^2 = a^2 + b^2$ if and only if $a = 0$ or $b = 0$.

SOLUTION Note that

$$(a+b)^2 = a^2 + 2ab + b^2.$$

Thus $(a+b)^2 = a^2 + b^2$ if and only if $2ab = 0$, which happens if and only if $a = 0$ or $b = 0$.

53. Show that $(a-1)^2 = a^2 - 1$ if and only if $a = 1$.

SOLUTION Note that

$$(a-1)^2 = a^2 - 2a + 1.$$

Thus $(a-1)^2 = a^2 - 1$ if and only if $-2a + 1 = -1$, which happens if and only if $a = 1$.

54. Explain why $(a-b)^2 = a^2 - b^2$ if and only if $b = 0$ or $b = a$.

SOLUTION Note that

$$(a-b)^2 = a^2 - 2ab + b^2.$$

Thus $(a-b)^2 = a^2 - b^2$ if and only if $-2ab + b^2 = -b^2$, which happens if and only if $2b(b-a) = 0$, which happens if and only if $b = 0$ or $b = a$.

55. Explain how you could show that $51 \times 49 = 2499$ in your head by using the identity $(a+b)(a-b) = a^2 - b^2$.

SOLUTION We have

$$\begin{aligned} 51 \times 49 &= (50+1)(50-1) \\ &= 50^2 - 1^2 \\ &= 2500 - 1 \\ &= 2499. \end{aligned}$$

56. Show that

$$\begin{aligned} &a^3 + b^3 + c^3 - 3abc \\ &\quad = (a+b+c)(a^2 + b^2 + c^2 - ab - bc - ac). \end{aligned}$$

SOLUTION The logical way to do a problem like this is to expand the right side of the equation above using the distributive property and hope for lots of cancellation. Fortunately that procedure works in this case:

$$\begin{aligned}(a+b+c)&(a^2+b^2+c^2-ab-bc-ac)\\&=a(a^2+b^2+c^2-ab-bc-ac)\\&\quad+b(a^2+b^2+c^2-ab-bc-ac)\\&\quad+c(a^2+b^2+c^2-ab-bc-ac)\\&=a^3+ab^2+ac^2-a^2b-abc-a^2c\\&\quad+a^2b+b^3+bc^2-ab^2-b^2c-abc\\&\quad+a^2c+b^2c+c^3-abc-bc^2-ac^2\\&=a^3+b^3+c^3-3abc\end{aligned}$$

57. Give an example to show that division does not satisfy the associative property.

SOLUTION Almost any random choice of three numbers will show that division does not satisfy the associative property. For example,

$$(2/3)/5=\frac{2}{3}\cdot\frac{1}{5}=\frac{2}{15},$$

but

$$2/(3/5)=2\cdot\frac{5}{3}=\frac{10}{3}.$$

Because $\frac{2}{15}\neq\frac{10}{3}$, division does not satisfy the associative property.

58. The sales tax in San Francisco is 8.5%. Diners in San Francisco often compute a 17% tip on their before-tax restaurant bill by simply doubling the sales tax. For example, a \$64 dollar food and drink bill would come with a sales tax of \$5.44; doubling that amount would lead to a 17% tip of \$10.88 (which might be rounded up to \$11). Explain why this technique is an application of the associativity of multiplication.

SOLUTION Let b denote the food and drink bill. Thus the 8.5% sales tax equals $0.85b$, and doubling the sales tax gives $2(0.85b)$.

A 17% tip would equal $0.17b$. Thus the technique described in this problem requires that $2(0.85b)$ equals $0.17b$. This equality indeed follows from the associativity of multiplication because

$$2(0.85b)=(2\cdot 0.85)b=0.17b.$$

59. A quick way to compute a 15% tip on a restaurant bill is first to compute 10% of the bill (by shifting the decimal point) and then add half of that amount for the total tip. For example, 15%

of a \$43 restaurant bill is \$4.30 + \$2.15, which equals \$6.45. Explain why this technique is an application of the distributive property.

SOLUTION Let b denote the food and drink bill. Thus 10% of the bill equals $0.1b$, half that amount equals $\frac{0.1b}{2}$, and the sum of those two amounts equals $0.1b + \frac{0.1b}{2}$.

A 15% tip would equal $0.15b$. Thus the technique described in the problem requires that $0.1b + \frac{0.1b}{2}$ equals $0.15b$. This equality indeed follows from the distributive property because

$$\begin{aligned} 0.1b + \tfrac{0.1b}{2} &= (0.1 + \tfrac{0.1}{2})b \\ &= (0.1 + 0.05)b \\ &= 0.15b. \end{aligned}$$

60. Suppose $b \neq 0$ and $d \neq 0$. Explain why

$$\frac{a}{b} = \frac{c}{d} \quad \text{if and only if} \quad ad = bc.$$

SOLUTION First suppose that

$$\frac{a}{b} = \frac{c}{d}.$$

Multiply both sides of the equation above by bd to conclude that $ad = bc$.

Now suppose that

$$ad = bc.$$

Divide both sides of the equation above by bd to conclude that $\frac{a}{b} = \frac{c}{d}$.

61. The first letters of the phrase "Please excuse my dear Aunt Sally" are used by some people to remember the order of operations: parentheses, exponents (which we will discuss in a later chapter), multiplication, division, addition, subtraction. Make up a catchy phrase that serves the same purpose but with exponents excluded.

SOLUTION An unimaginative modification of the phrase above to exclude exponents would be "Pardon my dear Aunt Sally." Another possibility is "Problems may demand a solution." No doubt students will come up with more clever possibilities.

62. (a) Verify that

$$\frac{16}{2} - \frac{25}{5} = \frac{16 - 25}{2 - 5}.$$

(b) From the example above you may be tempted to think that

$$\frac{a}{b} - \frac{c}{d} = \frac{a - c}{b - d}$$

provided none of the denominators equals 0. Give an example to show that this is not true.

SOLUTION

(a) We have

$$\frac{16}{2} - \frac{25}{5} = 8 - 5 = 3$$

and

$$\frac{16-25}{2-5} = \frac{-9}{-3} = 3.$$

Thus

$$\frac{16}{2} - \frac{25}{5} = \frac{16-25}{2-5}.$$

(b) Almost any random choices of a, b, c, and d will provide the requested example. One simple possibility is to take $a = b = 1$ and $c = d = 2$. Then

$$\frac{a}{b} - \frac{c}{d} = \frac{1}{1} - \frac{2}{2} = 1 - 1 = 0$$

and

$$\frac{a-c}{b-d} = \frac{1-2}{1-2} = \frac{-1}{-1} = 1.$$

Thus for these values of a, b, c, and d we have

$$\frac{a}{b} - \frac{c}{d} \neq \frac{a-c}{b-d}.$$

63. Suppose $b \neq 0$ and $d \neq 0$. Explain why

$$\frac{a}{b} - \frac{c}{d} = \frac{ad-bc}{bd}.$$

SOLUTION

$$\begin{aligned}\frac{a}{b} - \frac{c}{d} &= \frac{a}{b} \cdot \frac{d}{d} - \frac{b}{b} \cdot \frac{c}{d} \\ &= \frac{ad-bc}{bd}\end{aligned}$$

Solutions to Exercises, Section 1.3

1. Evaluate $|-4| + |4|$.

   SOLUTION $|-4| + |4| = 4 + 4 = 8$

2. Evaluate $|5| + |-6|$.

   SOLUTION $|5| + |-6| = 5 + 6 = 11$

3. Find all numbers with absolute value 9.

   SOLUTION The only numbers whose absolute value equals 9 are 9 and -9.

4. Find all numbers with absolute value 10.

   SOLUTION The only numbers whose absolute value equals 10 are 10 and -10.

In Exercises 5–18, find all numbers x satisfying the given equation.

5. $|2x - 6| = 11$

   SOLUTION The equation $|2x - 6| = 11$ implies that $2x - 6 = 11$ or $2x - 6 = -11$. Solving these equations for x gives $x = \frac{17}{2}$ or $x = -\frac{5}{2}$.

6. $|5x + 8| = 19$

   SOLUTION The equation $|5x + 8| = 19$ implies that $5x + 8 = 19$ or $5x + 8 = -19$. Solving these equations for x gives $x = \frac{11}{5}$ or $x = -\frac{27}{5}$.

7. $\left|\frac{x+1}{x-1}\right| = 2$

   SOLUTION The equation $\left|\frac{x+1}{x-1}\right| = 2$ implies that $\frac{x+1}{x-1} = 2$ or $\frac{x+1}{x-1} = -2$. Solving these equations for x gives $x = 3$ or $x = \frac{1}{3}$.

8. $\left|\frac{3x+2}{x-4}\right| = 5$

   SOLUTION The equation $\left|\frac{3x+2}{x-4}\right| = 5$ implies that $\frac{3x+2}{x-4} = 5$ or $\frac{3x+2}{x-4} = -5$. Solving these equations for x gives $x = 11$ or $x = \frac{9}{4}$.

9. $|x-3| + |x-4| = 9$

   SOLUTION First, consider numbers x such that $x > 4$. In this case, we have $x - 3 > 0$ and $x - 4 > 0$, which implies that $|x - 3| = x - 3$ and $|x - 4| = x - 4$. Thus the original equation becomes

   $$x - 3 + x - 4 = 9,$$

   which can be rewritten as $2x - 7 = 9$, which can easily be solved to yield $x = 8$. Substituting 8 for x in the original equation shows that $x = 8$ is indeed a solution (make sure you do this check).

   Second, consider numbers x such that $x < 3$. In this case, we have $x - 3 < 0$ and $x - 4 < 0$, which implies that $|x - 3| = 3 - x$ and $|x - 4| = 4 - x$. Thus the original equation becomes

$$3 - x + 4 - x = 9,$$

which can be rewritten as $7 - 2x = 9$, which can easily be solved to yield $x = -1$. Substituting -1 for x in the original equation shows that $x = -1$ is indeed a solution (make sure you do this check).

Third, we need to consider the only remaining possibility, which is that $3 \le x \le 4$. In this case, we have $x - 3 \ge 0$ and $x - 4 \le 0$, which implies that $|x - 3| = x - 3$ and $|x - 4| = 4 - x$. Thus the original equation becomes

$$x - 3 + 4 - x = 9,$$

which can be rewritten as $1 = 9$, which holds for no values of x.

Thus we can conclude that 8 and -1 are the only values of x that satisfy the original equation.

10. $|x+1| + |x-2| = 7$

SOLUTION First, consider numbers x such that $x > 2$. In this case, we have $x + 1 > 0$ and $x - 2 > 0$, which implies that $|x + 1| = x + 1$ and $|x - 2| = x - 2$. Thus the original equation becomes

$$x + 1 + x - 2 = 7,$$

which can be rewritten as $2x - 1 = 7$, which can easily be solved to yield $x = 4$. Substituting 4 for x in the original equation shows that $x = 4$ is indeed a solution (make sure you do this check).

Second, consider numbers x such that $x < -1$. In this case, we have $x + 1 < 0$ and $x - 2 < 0$, which implies that $|x + 1| = -x - 1$ and $|x - 2| = 2 - x$. Thus the original equation becomes

$$-x - 1 + 2 - x = 7,$$

which can be rewritten as $1 - 2x = 7$, which can easily be solved to yield $x = -3$. Substituting -3 for x in the original equation shows that $x = -3$ is indeed a solution (make sure you do this check).

Third, we need to consider the only remaining possibility, which is that $-1 \le x \le 2$. In this case, we have $x + 1 \ge 0$ and $x - 2 \le 0$, which implies that $|x + 1| = x + 1$ and $|x - 2| = 2 - x$. Thus the original equation becomes

$$x + 1 + 2 - x = 7,$$

which can be rewritten as $3 = 7$, which holds for no values of x.

Thus we can conclude that 4 and -3 are the only values of x that satisfy the original equation.

11. $|x-3| + |x-4| = 1$

SOLUTION If $x > 4$, then the distance from x to 3 is bigger than 1, and thus $|x - 3| > 1$ and thus $|x - 3| + |x - 4| > 1$. Hence there are no solutions to the equation above with $x > 4$.

If $x < 3$, then the distance from x to 4 is bigger than 1, and thus $|x - 4| > 1$ and thus $|x - 3| + |x - 4| > 1$. Hence there are no solutions to the equation above with $x < 3$.

The only remaining possibility is that $3 \le x \le 4$. In this case, we have $x - 3 \ge 0$ and $x - 4 \le 0$, which implies that $|x - 3| = x - 3$ and $|x - 4| = 4 - x$, which implies that

$$|x-3|+|x-4|=(x-3)+(4-x)=1.$$

Thus the set of numbers x such that $|x-3|+|x-4|=1$ is the interval $[3,4]$.

12. $|x+1|+|x-2|=3$

SOLUTION If $x>2$, then the distance from x to -1 is bigger than 3, and thus $|x+1|>3$ and thus $|x+1|+|x-2|>3$. Hence there are no solutions to the equation above with $x>2$.

If $x<-1$, then the distance from x to 2 is bigger than 3, and thus $|x-2|>3$ and thus $|x+1|+|x-2|>3$. Hence there are no solutions to the equation above with $x<-1$.

The only remaining possibility is that $-1 \le x \le 2$. In this case, we have $x+1 \ge 0$ and $x-2 \le 0$, which implies that $|x+1|=x+1$ and $|x-2|=2-x$, which implies that

$$|x+1|+|x-2|=(x+1)+(2-x)=3.$$

Thus the set of numbers x such that $|x+1|+|x-2|=3$ is the interval $[-1,2]$.

13. $|x-3|+|x-4|=\frac{1}{2}$

SOLUTION As we saw in the solution to Exercise 11, if $x>4$ or $x<3$, then $|x-3|+|x-4|>1$, and in particular $|x-3|+|x-4| \ne \frac{1}{2}$.

We also saw in the solution to Exercise 11 that if $3 \le x \le 4$, then $|x-3|+|x-4|=1$, and in particular $|x-3|+|x-4| \ne \frac{1}{2}$.

Thus there are no numbers x such that $|x-3|+|x-4|=\frac{1}{2}$.

14. $|x+1|+|x-2|=2$

SOLUTION As we saw in the solution to Exercise 12, if $x>2$ or $x<-1$, then $|x+1|+|x-2|>3$, and in particular $|x+1|+|x-2| \ne 2$.

We also saw in the solution to Exercise 12 that if $-1 \le x \le 2$, then $|x+1|+|x-2|=3$, and in particular $|x+1|+|x-2| \ne 2$.

Thus there are no numbers x such that $|x+1|+|x-2|=2$.

15. $|x+3|=x+3$

SOLUTION Note that $|x+3|=x+3$ if and only if $x+3 \ge 0$, which is equivalent to the inequality $x \ge -3$. Thus the set of numbers x such that $|x+3|=x+3$ is the interval $[-3,\infty)$.

16. $|x-5|=5-x$

SOLUTION Note that $|x-5|=5-x$ if and only if $x-5 \le 0$, which is equivalent to the inequality $x \le 5$. Thus the set of numbers x such that $|x-5|=5-x$ is the interval $(-\infty,5]$.

17. $|x|=x+1$

SOLUTION If $x \ge 0$, then $|x|=x$ and the equation above becomes the equation $x=x+1$, which has no solutions.

If $x < 0$, then $|x| = -x$ and the equation above becomes the equation $-x = x + 1$, which has the solution $x = -\frac{1}{2}$. Substituting $-\frac{1}{2}$ for x in the equation above shows that $x = -\frac{1}{2}$ is indeed a solution to the equation.

Thus the only number x satisfying $|x| = x + 1$ is $-\frac{1}{2}$.

18. $|x + 3| = x + 5$

SOLUTION If $x \geq -3$, then $x + 3 \geq 0$ and thus $|x + 3| = x + 3$ and the equation above becomes the equation $x + 3 = x + 5$, which has no solutions.

If $x < -3$, then $|x + 3| = -x - 3$ and the equation above becomes the equation $-x - 3 = x + 5$, which has the solution $x = -4$. Substituting -4 for x in the equation above shows that $x = -4$ is indeed a solution to the equation.

Thus the only number x satisfying $|x + 3| = x + 5$ is -4.

In Exercises 19–28, write each union as a single interval.

19. $[2, 7) \cup [5, 20)$

SOLUTION The first interval is $\{x : 2 \leq x < 7\}$, which includes the left endpoint 2 but does not include the right endpoint 7. The second interval is $\{x : 5 \leq x < 20\}$, which includes the left endpoint 5 but does not include the right endpoint 20. The set of numbers in at least one of these sets equals $\{x : 2 \leq x < 20\}$, as can be seen below:

Thus $[2, 7) \cup [5, 20) = [2, 20)$.

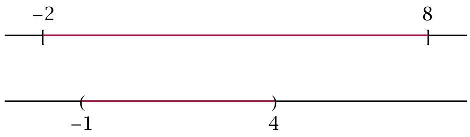
20. $[-8, -3) \cup [-6, -1)$

SOLUTION The first interval is the set $\{x : -8 \leq x < -3\}$, which includes the left endpoint -8 but does not include the right endpoint -3. The second interval is the set $\{x : -6 \leq x < -1\}$, which includes the left endpoint -6 but does not include the right endpoint -1. The set of numbers that are in at least one of these sets equals $\{x : -8 \leq x < -1\}$, as can be seen below:

Thus $[-8, -3) \cup [-6, -1) = [-8, -1)$.

21. $[-2, 8] \cup (-1, 4)$

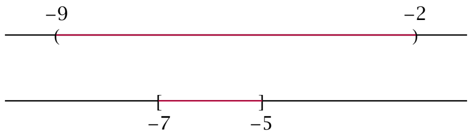
SOLUTION The first interval is the set $\{x : -2 \le x \le 8\}$, which includes both endpoints. The second interval is $\{x : -1 < x < 4\}$, which does not include either endpoint. The set of numbers in at least one of these sets equals $\{x : -2 \le x \le 8\}$, as can be seen below:



Thus $[-2, 8] \cup (-1, 4) = [-2, 8]$.

22. $(-9, -2) \cup [-7, -5]$

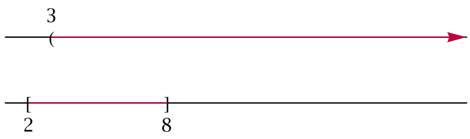
SOLUTION The first interval is the set $\{x : -9 < x < -2\}$, which does not include either endpoint. The second interval is the set $\{x : -7 \le x \le -5\}$, which includes both endpoints. The set of numbers that are in at least one of these sets equals $\{x : -9 < x < -2\}$, as can be seen below:



Thus $(-9, -2) \cup [-7, -5] = (-9, -2)$.

23. $(3, \infty) \cup [2, 8]$

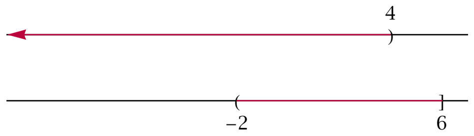
SOLUTION The first interval is $\{x : 3 < x\}$, which does not include the left endpoint and which has no right endpoint. The second interval is $\{x : 2 \le x \le 8\}$, which includes both endpoints. The set of numbers in at least one of these sets equals $\{x : 2 \le x\}$, as can be seen below:



Thus $(3, \infty) \cup [2, 8] = [2, \infty)$.

24. $(-\infty, 4) \cup (-2, 6]$

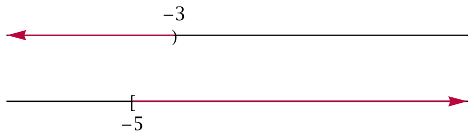
SOLUTION The first interval is the set $\{x : x < 4\}$, which has no left endpoint and which does not include the right endpoint. The second interval is the set $\{x : -2 < x \le 6\}$, which does not include the left endpoint but does include the right endpoint. The set of numbers that are in at least one of these sets equals $\{x : x \le 6\}$, as can be seen below:



Thus $(-\infty, 4) \cup (-2, 6] = (-\infty, 6]$.

25. $(-\infty, -3) \cup [-5, \infty)$

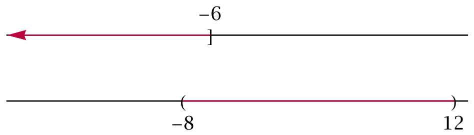
SOLUTION The first interval is $\{x : x < -3\}$, which has no left endpoint and which does not include the right endpoint. The second interval is $\{x : -5 \le x\}$, which includes the left endpoint and which has no right endpoint. The set of numbers in at least one of these sets equals the entire real line, as can be seen below:



Thus $(-\infty, -3) \cup [-5, \infty) = (-\infty, \infty)$.

26. $(-\infty, -6] \cup (-8, 12)$

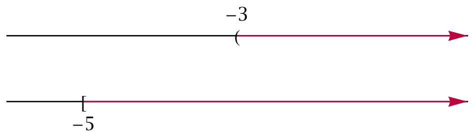
SOLUTION The first interval is the set $\{x : x \le -6\}$, which has no left endpoint and which includes the right endpoint. The second interval is the set $\{x : -8 < x < 12\}$, which does not include either endpoint. The set of numbers that are in at least one of these sets equals $\{x : x < 12\}$, as can be seen below:



Thus $(-\infty, -6] \cup (-8, 12) = (-\infty, 12)$.

27. $(-3, \infty) \cup [-5, \infty)$

SOLUTION The first interval is $\{x : -3 < x\}$, which does not include the left endpoint and which has no right endpoint. The second interval is $\{x : -5 \le x\}$, which includes the left endpoint and which has no right endpoint. The set of numbers that are in at least one of these sets equals $\{x : -5 \le x\}$, as can be seen below:



Thus $(-3, \infty) \cup [-5, \infty) = [-5, \infty)$.

28. $(-\infty, -10] \cup (-\infty, -8]$

SOLUTION The first interval is the set $\{x : x \leq -10\}$, which has no left endpoint and which includes the right endpoint. The second interval is the set $\{x : x \leq -8\}$, which has no left endpoint and which includes the right endpoint. The set of numbers that are in at least one of these sets equals $\{x : x \leq -8\}$, as can be seen below:

Thus $(-\infty, -10] \cup (-\infty, -8] = (-\infty, -8]$.

29. A medicine is known to decompose and become ineffective if its temperature ever reaches 103 degrees Fahrenheit or more. Write an interval to represent the temperatures (in degrees Fahrenheit) at which the medicine is ineffective.

SOLUTION The medicine is ineffective at all temperatures of 103 degrees Fahrenheit or greater, which corresponds to the interval $[103, \infty)$.

30. At normal atmospheric pressure, water boils at all temperatures of 100 degrees Celsius and higher. Write an interval to represent the temperatures (in degrees Celsius) at which water boils.

SOLUTION Water boils at all temperatures of 100 degrees Celsius or greater, which corresponds to the interval $[100, \infty)$.

31. A shoelace manufacturer guarantees that its 33-inch shoelaces will be 33 inches long, with an error of at most 0.1 inch.

(a) Write an inequality using absolute values and the length s of a shoelace that gives the condition that the shoelace does not meet the guarantee.

(b) Write the set of numbers satisfying the inequality in part (a) as a union of two intervals.

SOLUTION

(a) The error in the shoelace length is $|s - 33|$. Thus a shoelace with length s does not meet the guarantee if $|s - 33| > 0.1$.

(b) Because $33 - 0.1 = 32.9$ and $33 + 0.1 = 33.1$, the set of numbers s such that $|s - 33| > 0.1$ is $(-\infty, 32.9) \cup (33.1, \infty)$.

32. A copying machine works with paper that is 8.5 inches wide, provided that the error in the paper width is less than 0.06 inch.

(a) Write an inequality using absolute values and the length w of the paper that gives the condition that the paper's width fails the requirements of the copying machine.

(b) Write the set of numbers satisfying the inequality in part (a) as a union of two intervals.

SOLUTION

(a) The error in the paper width is $|w - 8.5|$. Thus paper with width w fails the requirements of the copying machine if $|w - 8.5| \geq 0.06$.

(b) Because $8.5 - 0.06 = 8.44$ and $8.5 + 0.06 = 8.56$, the set of numbers w such that $|w - 8.5| \geq 0.06$ is $(-\infty, 8.44] \cup [8.56, \infty)$.

33. Give four examples of pairs of real numbers a and b such that

$$|a + b| = 2 \quad \text{and} \quad |a| + |b| = 8.$$

SOLUTION First consider the case where $a \geq 0$ and $b \geq 0$. In this case, we have $a + b \geq 0$. Thus the equations above become

$$a + b = 2 \quad \text{and} \quad a + b = 8.$$

There are no solutions to the simultaneous equations above, because $a + b$ cannot simultaneously equal both 2 and 8.

Next consider the case where $a < 0$ and $b < 0$. In this case, we have $a + b < 0$. Thus the equations above become

$$-a - b = 2 \quad \text{and} \quad -a - b = 8.$$

There are no solutions to the simultaneous equations above, because $-a - b$ cannot simultaneously equal both 2 and 8.

Now consider the case where $a \geq 0$, $b < 0$, and $a + b \geq 0$. In this case the equations above become

$$a + b = 2 \quad \text{and} \quad a - b = 8.$$

Solving these equations for a and b, we get $a = 5$ and $b = -3$.

Now consider the case where $a \geq 0$, $b < 0$, and $a + b < 0$. In this case the equations above become

$$-a - b = 2 \quad \text{and} \quad a - b = 8.$$

Solving these equations for a and b, we get $a = 3$ and $b = -5$.

Now consider the case where $a < 0$, $b \geq 0$, and $a + b \geq 0$. In this case the equations above become

$$a + b = 2 \quad \text{and} \quad -a + b = 8.$$

Solving these equations for a and b, we get $a = -3$ and $b = 5$.

Now consider the case where $a < 0$, $b \geq 0$, and $a + b < 0$. In this case the equations above become

$$-a - b = 2 \quad \text{and} \quad -a + b = 8.$$

Solving these equations for a and b, we get $a = -5$ and $b = 3$.

At this point, we have considered all possible cases. Thus the only solutions are $a = 5$, $b = -3$, or $a = 3$, $b = -5$, or $a = -3$, $b = 5$, or $a = -5$, $b = 3$.

34. Give four examples of pairs of real numbers a and b such that

$$|a + b| = 3 \quad \text{and} \quad |a| + |b| = 11.$$

SOLUTION First consider the case where $a \geq 0$ and $b \geq 0$. In this case, we have $a + b \geq 0$. Thus the equations above become

$$a + b = 3 \quad \text{and} \quad a + b = 11.$$

There are no solutions to the simultaneous equations above, because $a + b$ cannot simultaneously equal both 3 and 11.

Next consider the case where $a < 0$ and $b < 0$. In this case, we have $a + b < 0$. Thus the equations above become

$$-a - b = 3 \quad \text{and} \quad -a - b = 11.$$

There are no solutions to the simultaneous equations above, because $-a - b$ cannot simultaneously equal both 3 and 11.

Now consider the case where $a \geq 0$, $b < 0$, and $a + b \geq 0$. In this case the equations above become

$$a + b = 3 \quad \text{and} \quad a - b = 11.$$

Solving these equations for a and b, we get $a = 7$ and $b = -4$.

Now consider the case where $a \geq 0$, $b < 0$, and $a + b < 0$. In this case the equations above become

$$-a - b = 3 \quad \text{and} \quad a - b = 11.$$

Solving these equations for a and b, we get $a = 4$ and $b = -7$.

Now consider the case where $a < 0$, $b \geq 0$, and $a + b \geq 0$. In this case the equations above become

$$a + b = 3 \quad \text{and} \quad -a + b = 11.$$

Solving these equations for a and b, we get $a = -4$ and $b = 7$.

Now consider the case where $a < 0$, $b \geq 0$, and $a + b < 0$. In this case the equations above become

$$-a - b = 3 \quad \text{and} \quad -a + b = 11.$$

Solving these equations for a and b, we get $a = -7$ and $b = 4$.

At this point, we have considered all possible cases. Thus the only solutions are $a = 7$, $b = -4$, or $a = 4$, $b = -7$, or $a = -4$, $b = 7$, or $a = -7$, $b = 4$.

In Exercises 35–46, write each set as an interval or as a union of two intervals.

35. $\{x : |x - 4| < \frac{1}{10}\}$

SOLUTION The inequality $|x - 4| < \frac{1}{10}$ is equivalent to the inequality

$$-\tfrac{1}{10} < x - 4 < \tfrac{1}{10}.$$

Add 4 to all parts of this inequality, getting

$$4 - \tfrac{1}{10} < x < 4 + \tfrac{1}{10},$$

which is the same as

$$\tfrac{39}{10} < x < \tfrac{41}{10}.$$

Thus $\{x : |x - 4| < \frac{1}{10}\} = (\frac{39}{10}, \frac{41}{10})$.

36. $\{x : |x + 2| < \frac{1}{100}\}$

SOLUTION The inequality $|x + 2| < \frac{1}{100}$ is equivalent to the inequality

$$-\tfrac{1}{100} < x + 2 < \tfrac{1}{100}.$$

Add -2 to all parts of this inequality, getting

$$-2 - \tfrac{1}{100} < x < -2 + \tfrac{1}{100},$$

which is the same as

$$-\tfrac{201}{100} < x < -\tfrac{199}{100}.$$

Thus $\{x : |x + 2| < \frac{1}{100}\} = (-\frac{201}{100}, -\frac{199}{100})$.

37. $\{x : |x + 4| < \frac{\varepsilon}{2}\}$; here $\varepsilon > 0$
[*Mathematicians often use the Greek letter* ε, *which is called* **epsilon**, *to denote a small positive number.*]

SOLUTION The inequality $|x + 4| < \frac{\varepsilon}{2}$ is equivalent to the inequality

$$-\frac{\varepsilon}{2} < x + 4 < \frac{\varepsilon}{2}.$$

Add -4 to all parts of this inequality, getting

$$-4 - \frac{\varepsilon}{2} < x < -4 + \frac{\varepsilon}{2}.$$

Thus $\{x : |x + 4| < \frac{\varepsilon}{2}\} = (-4 - \frac{\varepsilon}{2}, -4 + \frac{\varepsilon}{2})$.

38. $\{x : |x - 2| < \frac{\varepsilon}{3}\}$; here $\varepsilon > 0$

SOLUTION The inequality $|x - 2| < \frac{\varepsilon}{3}$ is equivalent to the inequality

$$-\frac{\varepsilon}{3} < x - 2 < \frac{\varepsilon}{3}.$$

Add 2 to all parts of this inequality, getting

$$2 - \frac{\varepsilon}{3} < x < 2 + \frac{\varepsilon}{3}.$$

Thus $\{x : |x - 2| < \frac{\varepsilon}{3}\} = (2 - \frac{\varepsilon}{3}, 2 + \frac{\varepsilon}{3})$.

39. $\{y : |y - a| < \varepsilon\}$; here $\varepsilon > 0$

SOLUTION The inequality $|y - a| < \varepsilon$ is equivalent to the inequality

$$-\varepsilon < y - a < \varepsilon.$$

Add a to all parts of this inequality, getting

$$a - \varepsilon < y < a + \varepsilon.$$

Thus $\{y : |y - a| < \varepsilon\} = (a - \varepsilon, a + \varepsilon)$.

40. $\{y : |y + b| < \varepsilon\}$; here $\varepsilon > 0$

SOLUTION The inequality $|y + b| < \varepsilon$ is equivalent to the inequality

$$-\varepsilon < y + b < \varepsilon.$$

Add $-b$ to all parts of this inequality, getting

$$-b - \varepsilon < y < -b + \varepsilon.$$

Thus $\{y : |y + b| < \varepsilon\} = (-b - \varepsilon, -b + \varepsilon)$.

41. $\{x : |3x - 2| < \frac{1}{4}\}$

SOLUTION The inequality $|3x - 2| < \frac{1}{4}$ is equivalent to the inequality

$$-\tfrac{1}{4} < 3x - 2 < \tfrac{1}{4}.$$

Add 2 to all parts of this inequality, getting

$$\tfrac{7}{4} < 3x < \tfrac{9}{4}.$$

Now divide all parts of this inequality by 3, getting

$$\tfrac{7}{12} < x < \tfrac{3}{4}.$$

Thus $\{x : |3x - 2| < \frac{1}{4}\} = (\frac{7}{12}, \frac{3}{4})$.

42. $\{x : |4x - 3| < \frac{1}{5}\}$

SOLUTION The inequality $|4x - 3| < \frac{1}{5}$ is equivalent to the inequality

$$-\tfrac{1}{5} < 4x - 3 < \tfrac{1}{5}.$$

Add 3 to all parts of this inequality, getting

$$\tfrac{14}{5} < 4x < \tfrac{16}{5}.$$

Now divide all parts of this inequality by 4, getting

$$\tfrac{7}{10} < x < \tfrac{4}{5}.$$

Thus $\{x : |4x - 3| < \frac{1}{5}\} = (\frac{7}{10}, \frac{4}{5})$.

43. $\{x : |x| > 2\}$

SOLUTION The inequality $|x| > 2$ means that $x > 2$ or $x < -2$. Thus $\{x : |x| > 2\} = (-\infty, -2) \cup (2, \infty)$.

44. $\{x : |x| > 9\}$

SOLUTION The inequality $|x| > 9$ means that $x > 9$ or $x < -9$. Thus $\{x : |x| > 9\} = (-\infty, -9) \cup (9, \infty)$.

45. $\{x : |x - 5| \geq 3\}$

SOLUTION The inequality $|x - 5| \geq 3$ means that $x - 5 \geq 3$ or $x - 5 \leq -3$. Adding 5 to both sides of these equalities shows that $x \geq 8$ or $x \leq 2$. Thus $\{x : |x - 5| \geq 3\} = (-\infty, 2] \cup [8, \infty)$.

46. $\{x : |x + 6| \geq 2\}$

SOLUTION The inequality $|x + 6| \geq 2$ means that $x + 6 \geq 2$ or $x + 6 \leq -2$. Subtracting 6 from both sides of these equalities shows that $x \geq -4$ or $x \leq -8$. Thus $\{x : |x + 6| \geq 2\} = (-\infty, -8] \cup [-4, \infty)$.

The* intersection *of two sets of numbers consists of all numbers that are in both sets. If A and B are sets, then their intersection is denoted by $A \cap B$. In Exercises 47–56, write each intersection as a single interval.

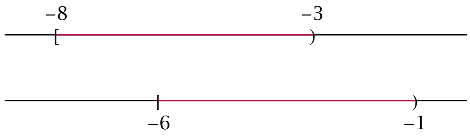
47. $[2, 7) \cap [5, 20)$

SOLUTION The first interval is the set $\{x : 2 \leq x < 7\}$, which includes the left endpoint 2 but does not include the right endpoint 7. The second interval is the set $\{x : 5 \leq x < 20\}$, which includes the left endpoint 5 but does not include the right endpoint 20. The set of numbers that are in both these sets equals $\{x : 5 \leq x < 7\}$, as can be seen below:

Thus $[2, 7) \cap [5, 20) = [5, 7)$.

48. $[-8, -3) \cap [-6, -1)$

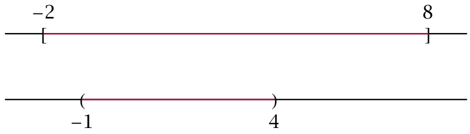
SOLUTION The first interval is the set $\{x : -8 \leq x < -3\}$, which includes the left endpoint -8 but does not include the right endpoint -3. The second interval is the set $\{x : -6 \leq x < -1\}$, which includes the left endpoint -6 but does not include the right endpoint -1. The set of numbers that are in both these sets equals $\{x : -6 \leq x < -3\}$, as can be seen below:



Thus $[-8,-3) \cap [-6,-1) = [-6,-3)$.

49. $[-2,8] \cap (-1,4)$

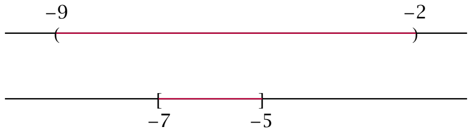
SOLUTION The first interval is the set $\{x : -2 \le x \le 8\}$, which includes both endpoints. The second interval is the set $\{x : -1 < x < 4\}$, which includes neither endpoint. The set of numbers that are in both these sets equals $\{x : -1 < x < 4\}$, as can be seen below:



Thus $[-2,8] \cap (-1,4) = (-1,4)$.

50. $(-9,-2) \cap [-7,-5]$

SOLUTION The first interval is the set $\{x : -9 < x < -2\}$, which does not include either endpoint. The second interval is the set $\{x : -7 \le x \le -5\}$, which includes both endpoints. The set of numbers that are in both these sets equals $\{x : -7 \le x \le -5\}$, as can be seen below:



Thus $(-9,-2) \cap [-7,-5] = [-7,-5]$.

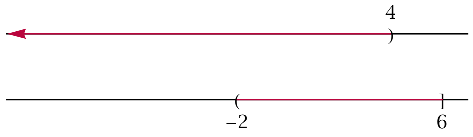
51. $(3,\infty) \cap [2,8]$

SOLUTION The first interval is $\{x : 3 < x\}$, which does not include the left endpoint and which has no right endpoint. The second interval is $\{x : 2 \le x \le 8\}$, which includes both endpoints. The set of numbers in both sets equals $\{x : 3 < x \le 8\}$, as can be seen below:

Thus $(3,\infty) \cap [2,8] = (3,8]$.

52. $(-\infty,4) \cap (-2,6]$

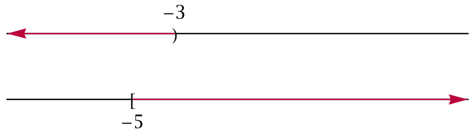
SOLUTION The first interval is the set $\{x : x < 4\}$, which has no left endpoint and which does not include the right endpoint. The second interval is the set $\{x : -2 < x \le 6\}$, which does not include the left endpoint but does include the right endpoint. The set of numbers that are in both these sets equals $\{x : -2 < x < 4\}$, as can be seen below:



Thus $(-\infty, 4) \cap (-2, 6] = (-2, 4)$.

53. $(-\infty, -3) \cap [-5, \infty)$

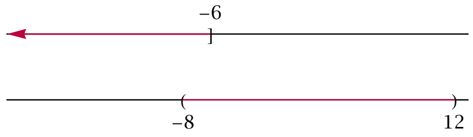
SOLUTION The first interval is $\{x : x < -3\}$, which has no left endpoint and which does not include the right endpoint. The second interval is $\{x : -5 \le x\}$, which includes the left endpoint and which has no right endpoint. The set of numbers in both sets equals $\{x : -5 \le x < -3\}$, as can be seen below:



Thus $(-\infty, -3) \cap [-5, \infty) = [-5, -3)$.

54. $(-\infty, -6] \cap (-8, 12)$

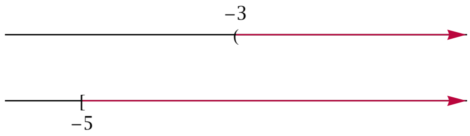
SOLUTION The first interval is the set $\{x : x \le -6\}$, which has no left endpoint and which includes the right endpoint. The second interval is the set $\{x : -8 < x < 12\}$, which includes neither endpoint. The set of numbers that are in both these sets equals $\{x : -8 < x \le -6\}$, as can be seen below:



Thus $(-\infty, -6] \cap (-8, 12) = (-8, -6]$.

55. $(-3, \infty) \cap [-5, \infty)$

SOLUTION The first interval is $\{x : -3 < x\}$, which does not include the left endpoint and which has no right endpoint. The second interval is $\{x : -5 \le x\}$, which includes the left endpoint and which has no right endpoint. The set of numbers in both sets equals $\{x : -3 < x\}$, as can be seen below:



Thus $(-3, \infty) \cap [-5, \infty) = (-3, \infty)$.

56. $(-\infty, -10] \cap (-\infty, -8]$

SOLUTION The first interval is the set $\{x : x \leq -10\}$, which has no left endpoint and which includes the right endpoint. The second interval is the set $\{x : x \leq -8\}$, which has no left endpoint and which includes the right endpoint. The set of numbers that are in both these sets equals $\{x : x \leq -10\}$, as can be seen below:

Thus $(-\infty, -10] \cap (-\infty, -8] = (-\infty, -10]$.

Solutions to Problems, Section 1.3

57. Suppose a and b are numbers. Explain why either $a < b$, $a = b$, or $a > b$.

SOLUTION Either a is left of b (in which case we have $a < b$) or $a = b$ or a is right of b (in which case we have $a > b$).

58. Show that if $a < b$ and $c \leq d$, then $a + c < b + d$.

SOLUTION Suppose $a < b$ and $c \leq d$. Then $b - a > 0$ and $d - c \geq 0$. This implies that $(b - a) + (d - c) > 0$, which can be rewritten as

$$(b + d) - (a + c) > 0,$$

which implies that $a + c < b + d$.

59. Show that if b is a positive number and $a < b$, then

$$\frac{a}{b} < \frac{a+1}{b+1}.$$

SOLUTION Suppose b is a positive number and $a < b$. Add ab to both sides of the inequality $a < b$, getting

$$ab + a < ab + b,$$

which can be rewritten as

$$a(b + 1) < b(a + 1).$$

Now multiply both sides of the inequality above by the positive number $\frac{1}{b(b+1)}$, getting

$$\frac{a}{b} < \frac{a+1}{b+1},$$

as desired.

60. In contrast to Problem 62 in Section 1.2, show that there do not exist positive numbers a, b, c, and d such that

$$\frac{a}{b} + \frac{c}{d} = \frac{a+c}{b+d}.$$

SOLUTION Suppose

$$\frac{a}{b} + \frac{c}{d} = \frac{a+c}{b+d}.$$

Multiplying both sides of the equation above by $bd(b + d)$ gives

$$ad(b + d) + cb(b + d) = (a + c)bd,$$

which can be rewritten as

$$abd + ad^2 + cb^2 + cbd = abd + cbd.$$

Subtracting $abd + cbd$ from both sides gives

$$ad^2 + cb^2 = 0.$$

However, the equation above cannot hold if a, b, c, and d are all positive numbers.

61. Explain why every open interval containing 0 contains an open interval centered at 0.

SOLUTION Suppose (a, b) is an open interval containing 0. This implies that $a < 0$ and $b > 0$. Let c be the minimum of the two numbers $|a|$ and b. Then the interval $(-c, c)$ is an open interval centered at 0 that is contained in the interval (a, b).

62. Give an example of an open interval and a closed interval whose union equals the interval $(2, 5)$.

SOLUTION Consider the open interval $(2, 5)$ and the closed interval $[3, 4]$. The union of these two intervals is the interval $(2, 5)$.

63. (a) True or false:

$$\text{If } a < b \text{ and } c < d, \text{ then } c - b < d - a.$$

(b) Explain your answer to part (a). This means that if the answer to part (a) is "true", then you should explain why $c - b < d - a$ whenever $a < b$ and $c < d$; if the answer to part (a) is "false", then you should give an example of numbers a, b, c, and d such that $a < b$ and $c < d$ but $c - b \geq d - a$.

SOLUTION

(a) True.

(b) Suppose $a < b$ and $c < d$. Because $a < b$, we have $-b < -a$. Adding the inequality $c < d$ to this, we get $c - b < d - a$.

64. (a) True or false:

$$\text{If } a < b \text{ and } c < d, \text{ then } ac < bd.$$

(b) Explain your answer to part (a). This means that if the answer to part (a) is "true", then you should explain why $ac < bd$ whenever $a < b$ and $c < d$; if the answer to part (a) is "false", then you should give an example of numbers a, b, c, and d such that $a < b$ and $c < d$ but $ac \geq bd$.

SOLUTION

(a) False.

(b) Take $a = -1$, $b = 0$, $c = -1$, $d = 0$. Then $a < b$ and $c < d$. However, $ac = 1$ and $bd = 0$, and thus $ac \geq bd$.

65. (a) True or false:

$$\text{If } 0 < a < b \text{ and } 0 < c < d, \text{ then } \frac{a}{d} < \frac{b}{c}.$$

(b) Explain your answer to part (a). This means that if the answer to part (a) is "true", then you should explain why $\frac{a}{d} < \frac{b}{c}$ whenever $0 < a < b$ and $0 < c < d$; if the answer to part (a) is "false", then you should give an example of numbers a, b, c, and d such that $0 < a < b$ and $0 < c < d$ but

$$\frac{a}{d} \geq \frac{b}{c}.$$

SOLUTION

(a) True.

(b) Suppose $0 < a < b$ and $0 < c < d$. Multiply both sides of the inequality $a < b$ by the positive number c, getting

$$ac < bc.$$

Now multiply both sides of the inequality $c < d$ by the positive number b, getting

$$bc < bd.$$

Now apply transitivity to the two inequalities displayed above to get

$$ac < bd.$$

Now multiply both sides of the inequality above by the positive number $\frac{1}{cd}$, getting

$$\frac{a}{d} < \frac{b}{c},$$

as desired.

66. Give an example of an open interval and a closed interval whose intersection equals the interval $(2, 5)$.

SOLUTION Consider the open interval $(2, 5)$ and the closed interval $[1, 6]$. The intersection of these two intervals is the interval $(2, 5)$.

67. Give an example of an open interval and a closed interval whose union equals the interval $[-3, 7]$.

SOLUTION Consider the open interval $(-2, 6)$ and the closed interval $[-3, 7]$. The union of these two intervals is the interval $[-3, 7]$.

68. Give an example of an open interval and a closed interval whose intersection equals the interval $[-3, 7]$.

SOLUTION Consider the open interval $(-4, 8)$ and the closed interval $[-3, 7]$. The intersection of these two intervals is the interval $[-3, 7]$.

69. Explain why the equation

$$|8x - 3| = -2$$

has no solutions.

SOLUTION No number has an absolute value that is negative. Thus there does not exist a number x such that the absolute value of $8x - 3$ equals -2. In other words, the equation $|8x - 3| = -2$ has no solutions.

70. Explain why

$$|a^2| = a^2$$

for every real number a.

SOLUTION Suppose a is a real number. Then $a^2 \geq 0$. Thus $|a^2| = a^2$.

71. Explain why

$$|ab| = |a||b|$$

for all real numbers a and b.

SOLUTION Let a and b be real numbers.

First consider the case where $a \geq 0$ and $b \geq 0$. Then $ab \geq 0$. Thus

$$|ab| = ab = |a||b|.$$

Next consider the case where $a \geq 0$ and $b < 0$. Then $ab \leq 0$. Thus

$$|ab| = -ab = a(-b) = |a||b|.$$

Next consider the case where $a < 0$ and $b \geq 0$. Then $ab \leq 0$. Thus

$$|ab| = -ab = (-a)b = |a||b|.$$

Finally consider the case where $a < 0$ and $b < 0$. Then $ab > 0$. Thus

$$|ab| = ab = (-a)(-b) = |a||b|.$$

We have found that in all possible cases, $|ab| = |a||b|$.

72. Explain why

$$|-a| = |a|$$

for all real numbers a.

SOLUTION Let a be a real number.

First consider the case where $a \geq 0$. Then $-a \leq 0$. Thus

$$|-a| = a = |a|.$$

Next consider the case where $a < 0$. Then $-a > 0$. Thus

$$|-a| = -a = |a|.$$

We have found that in all possible cases, $|-a| = |a|$.

73. Explain why

$$\left|\frac{a}{b}\right| = \frac{|a|}{|b|}$$

for all real numbers a and b (with $b \neq 0$).

SOLUTION Let a and b be real numbers with $b \neq 0$.

First consider the case where $a \geq 0$ and $b > 0$. Then $\frac{a}{b} \geq 0$. Thus

$$\left|\frac{a}{b}\right| = \frac{a}{b} = \frac{|a|}{|b|}.$$

Next consider the case where $a \geq 0$ and $b < 0$. Then $\frac{a}{b} \leq 0$. Thus

$$\left|\frac{a}{b}\right| = -\frac{a}{b} = \frac{a}{-b} = \frac{|a|}{|b|}.$$

Next consider the case where $a < 0$ and $b > 0$. Then $\frac{a}{b} < 0$. Thus

$$\left|\frac{a}{b}\right| = -\frac{a}{b} = \frac{-a}{b} = \frac{|a|}{|b|}.$$

Finally consider the case where $a < 0$ and $b < 0$. Then $\frac{a}{b} > 0$. Thus

$$\left|\frac{a}{b}\right| = \frac{a}{b} = \frac{-a}{-b} = \frac{|a|}{|b|}.$$

We have found that in all possible cases, $\left|\frac{a}{b}\right| = \frac{|a|}{|b|}$.

74. Give an example of a set of real numbers such that the average of any two numbers in the set is in the set, but the set is not an interval.

SOLUTION The set of rational numbers has the desired property. First, note that if b and c are rational numbers, then so is $b + c$ and hence $\frac{b+c}{2}$ (which is the average of b and c) is also a rational number. However, the set of rational numbers is not an interval because 1 and 2 are rational numbers but $\sqrt{2}$, which is between 1 and 2, is not a rational number.

75. (a) Show that if $a \geq 0$ and $b \geq 0$, then $|a + b| = |a| + |b|$.
(b) Show that if $a \geq 0$ and $b < 0$, then $|a + b| \leq |a| + |b|$.
(c) Show that if $a < 0$ and $b \geq 0$, then $|a + b| \leq |a| + |b|$.
(d) Show that if $a < 0$ and $b < 0$, then $|a + b| = |a| + |b|$.
(e) Explain why the previous four items imply that

$$|a + b| \leq |a| + |b|$$

for all real numbers a and b.

SOLUTION

(a) Suppose $a \geq 0$ and $b \geq 0$. Then $a + b \geq 0$. Thus

$$|a + b| = a + b = |a| + |b|.$$

(b) Suppose $a \geq 0$ and $b < 0$.
First consider the case where $a + b \geq 0$. Then

$$|a + b| = a + b = |a| - |b| \leq |a| + |b|.$$

Next consider the case where $a + b < 0$. Then

$$\begin{aligned} |a + b| &= -(a + b) \\ &= (-a) + (-b) \\ &= (-a) + |b| \\ &\leq |a| + |b|. \end{aligned}$$

We have found that in both possible cases concerning $a + b$, we have $|a + b| \leq |a| + |b|$.

(c) Interchanging the roles of a and b in part (b) and using the commutativity of addition shows that if $a < 0$ and $b \geq 0$, then $|a + b| \leq |a| + |b|$.

(d) Suppose $a < 0$ and $b < 0$. Then $a + b < 0$. Thus

$$|a + b| = -(a + b) = (-a) + (-b) = |a| + |b|.$$

(e) The four previous items cover all possible cases concerning whether $a \geq 0$ or $a < 0$ and $b \geq 0$ or $b < 0$. In all four cases we found that $|a + b| \leq |a| + |b|$ (note that the equation $|a + b| = |a| + |b|$ implies that $|a + b| \leq |a| + |b|$). Thus $|a + b| \leq |a| + |b|$ for all real numbers a and b.

76. Show that if a and b are real numbers such that

$$|a + b| < |a| + |b|,$$

then $ab < 0$.

SOLUTION Suppose a and b are real numbers such that $|a + b| < |a| + |b|$.

We cannot have $a = 0$ because then we would have

$$|a + b| = |b| = |a| + |b|.$$

We cannot have $b = 0$ because then we would have

$$|a + b| = |a| = |a| + |b|.$$

We cannot have $a > 0$ and $b > 0$, because then we would have $|a + b| = |a| + |b|$ by part (a) of the previous problem.

We cannot have $a < 0$ and $b < 0$, because then we would have $|a + b| = |a| + |b|$ by part (d) of the previous problem.

The only remaining possibilities are that $a > 0$ and $b < 0$ or $a < 0$ and $b > 0$. In both these cases, we have $ab < 0$, as desired.

77. Show that

$$\big||a| - |b|\big| \le |a - b|$$

for all real numbers a and b.

SOLUTION Note that

$$|a| = |(a - b) + b| \le |a - b| + |b|,$$

where the inequality above follows from part (e) of Problem 75. Subtracting $|b|$ from both sides of the inequality above gives

$$|a| - |b| \le |a - b|.$$

Interchanging the roles of a and b in the inequality above gives

$$|b| - |a| \le |b - a| = |a - b|.$$

Now $\big||a| - |b|\big|$ equals either $|a| - |b|$ or $|b| - |a|$. Either way, one of the two inequalities above implies that $\big||a| - |b|\big| \le |a - b|$, as desired.

Solutions to Chapter Review Questions, Chapter 1

1. Explain how the points on the real line correspond to the set of real numbers.

   SOLUTION Start with a horizontal line. Pick a point on this line an label it 0. Pick another point on the line to the right of 0 and label it 1. The distance between the point labeled 0 and the point labeled 1 becomes the unit of measurement.

   Each point to the right of 0 on the line corresponds to the distance (using the unit of measurement described above) between 0 and the point. Each point to the left of 0 on the line corresponds to the negative of the distance (using the unit of measurement) between 0 and the point.

2. Show that $7 - 6\sqrt{2}$ is an irrational number.

   SOLUTION Suppose $7 - 6\sqrt{2}$ is a rational number. Because

   $$6\sqrt{2} = 7 - (7 - 6\sqrt{2}),$$

   this implies that $6\sqrt{2}$ is the difference of two rational numbers, which implies that $6\sqrt{2}$ is a rational number. Because

   $$\sqrt{2} = \frac{6\sqrt{2}}{6},$$

   this implies that $\sqrt{2}$ is the quotient of two rational numbers, which implies that $\sqrt{2}$ is a rational number, which is not true. Thus our assumption that $7 - 6\sqrt{2}$ is a rational number must be false. In other words, $7 - 6\sqrt{2}$ is an irrational number.

3. What is the commutative property for addition?

   SOLUTION The commutative property for addition states that order does not matter in the sum of two numbers. In other words, $a + b = b + a$ for all real numbers a and b.

4. What is the commutative property for multiplication?

   SOLUTION The commutative property for multiplication states that order does not matter in the product of two numbers. In other words, $ab = ba$ for all real numbers a and b.

5. What is the associative property for addition?

   SOLUTION The associative property for addition states that grouping does not matter in the sum of three numbers. In other words, $(a + b) + c = a + (b + c)$ for all real numbers a, b, and c.

6. What is the associative property for multiplication?

   SOLUTION The associative property for multiplication states that grouping does not matter in the product of three numbers. In other words, $(ab)c = a(bc)$ for all real numbers a, b, and c.

7. Expand $(t + w)^2$.

   SOLUTION $(t + w)^2 = t^2 + 2tw + w^2$

8. Expand $(u - v)^2$.

SOLUTION $(u-v)^2 = u^2 - 2uv + v^2$

9. Expand $(x-y)(x+y)$.

SOLUTION $(x-y)(x+y) = x^2 - y^2$

10. Expand $(a+b)(x-y-z)$.

SOLUTION

$$\begin{aligned}(a+b)(x-y-z) &= a(x-y-z) + b(x-y-z)\\ &= ax - ay - az + bx - by - bz\end{aligned}$$

11. Expand $(a+b-c)^2$.

SOLUTION

$$\begin{aligned}(a+b-c)^2 &= (a+b-c)(a+b-c)\\ &= a(a+b-c) + b(a+b-c) - c(a+b-c)\\ &= a^2 + ab - ac + ab + b^2 - bc - ac - bc + c^2\\ &= a^2 + b^2 + c^2 + 2ab - 2ac - 2bc\end{aligned}$$

12. Simplify the expression $\dfrac{\frac{1}{t-b} - \frac{1}{t}}{b}$.

SOLUTION We start by evaluating the numerator:

$$\begin{aligned}\frac{1}{t-b} - \frac{1}{t} &= \frac{t}{t(t-b)} - \frac{t-b}{t(t-b)}\\ &= \frac{t-(t-b)}{t(t-b)}\\ &= \frac{b}{t(t-b)}.\end{aligned}$$

Thus

$$\frac{\frac{1}{t-b} - \frac{1}{t}}{b} = \frac{1}{t(t-b)}.$$

13. Find all real numbers x such that $|3x-4| = 5$.

SOLUTION The equation $|3x-4| = 5$ holds if and only if $3x-4 = 5$ or $3x-4 = -5$. Solving the equation $3x-4 = 5$ gives $x = 3$; solving the equation $3x-4 = -5$ gives $x = -\frac{1}{3}$. Thus the solutions to the equation $|3x-4| = 5$ are $x = 3$ and $x = -\frac{1}{3}$.

14. Give an example of two numbers x and y such that $|x + y|$ does not equal $|x| + |y|$.

SOLUTION As one example, take $x = 1$ and $y = -1$. Then

$$|x + y| = 0 \quad \text{but} \quad |x| + |y| = 2.$$

As another example, take $x = 2$ and $y = -1$. Then

$$|x + y| = 1 \quad \text{but} \quad |x| + |y| = 3.$$

15. Suppose $0 < a < b$ and $0 < c < d$. Explain why $ac < bd$.

SOLUTION Because $c > 0$, we can multiply both sides of the inequality $a < b$ by c to obtain

$$ac < bc.$$

Because $b > 0$, we can multiply both sides of the inequality $c < d$ to obtain

$$bc < bd.$$

Using transitivity and the two inequalities above, we have $ac < bd$, as desired.

16. Write the set $\{t : |t - 3| < \frac{1}{4}\}$ as an interval.

SOLUTION The inequality $|t - 3| < \frac{1}{4}$ is equivalent to the inequality

$$-\tfrac{1}{4} < t - 3 < \tfrac{1}{4}.$$

Add 3 to all parts of this inequality, getting

$$\tfrac{11}{4} < t < \tfrac{13}{4}.$$

Thus $\{t : |t - 3| < \frac{1}{4}\} = (\frac{11}{4}, \frac{13}{4})$.

17. Write the set $\{w : |5w + 2| < \frac{1}{3}\}$ as an interval.

SOLUTION The inequality $|5w + 2| < \frac{1}{3}$ is equivalent to the inequality

$$-\tfrac{1}{3} < 5w + 2 < \tfrac{1}{3}.$$

Add -2 to all parts of this inequality, getting

$$-\tfrac{7}{3} < 5w < -\tfrac{5}{3}.$$

Now divide all parts of this inequality by 5, getting

$$-\tfrac{7}{15} < w < -\tfrac{1}{3}.$$

Thus $\{w : |5w + 2| < \frac{1}{3}\} = (-\frac{7}{15}, -\frac{1}{3})$.

18. Explain why the sets $\{x : |8x - 5| < 2\}$ and $\{t : |5 - 8t| < 2\}$ are the same set.

SOLUTION First, note that in the description of the set $\{x : |8x - 5| < 2\}$, the variable x can be changed to any other variable (for example t) without changing the set. In other words,

$$\{x : |8x - 5| < 2\} = \{t : |8t - 5| < 2\}.$$

Second, note that $|8t - 5| = |5 - 8t|$. Thus

$$\{t : |8t - 5| < 2\} = \{t : |5 - 8t| < 2\}.$$

Putting together the two displayed equalities, we have

$$\{x : |8x - 5| < 2\} = \{t : |5 - 8t| < 2\},$$

as desired.

[*Students who have difficulty understanding the solution above may be convinced that it is correct by showing that both sets equal the interval* $(\frac{3}{8}, \frac{7}{8})$. *Calculating the intervals for both sets may be mathematically inefficient, but it may help show some students that the name of the variable is irrelevant.*]

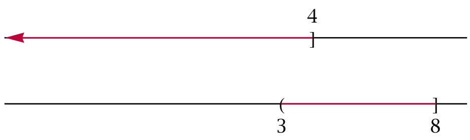
19. Write $[-5, 6) \cup [-1, 9)$ as an interval.

SOLUTION The first interval is the set $\{x : -5 \le x < 6\}$, which includes the left endpoint -5 but does not include the right endpoint 6. The second interval is the set $\{x : -1 \le x < 9\}$, which includes the left endpoint -1 but does not include the right endpoint 9. The set of numbers that are in at least one of these sets equals $\{x : -5 \le x < 9\}$, as can be seen below:

Thus $[-5, 6) \cup [-1, 9) = [-5, 9)$.

20. Write $(-\infty, 4] \cup (3, 8]$ as an interval.

SOLUTION The first interval is the set $\{x : x \le 4\}$, which has no left endpoint and which includes the right endpoint 4. The second interval is the set $\{x : 3 < x \le 8\}$, which does not include the left endpoint 3 but does include the right endpoint 8. The set of numbers that are in at least one of these sets equals $\{x : x \le 8\}$, as can be seen below:



Thus $(-\infty, 4] \cup (3, 8) = (-\infty, 8]$.

21. Explain why $[7, \infty]$ is not an interval of real numbers.

SOLUTION The symbol ∞ does not represent a real number. The closed brackets in the notation $[7, \infty]$ indicate that both endpoints should be included. However, because ∞ is not a real number, the notation $[7, \infty]$ makes no sense as an interval of real numbers.

22. Write the set $\{t : |2t + 7| \geq 5\}$ as a union of two intervals.

SOLUTION The inequality $|2t + 7| \geq 5$ means that $2t + 7 \geq 5$ or $2t + 7 \leq -5$. Adding -7 to both sides of these inequalities shows that $2t \geq -2$ or $2t \leq -12$. Dividing both inequalities by 2 shows that $t \geq -1$ or $t \leq -6$. Thus $\{t : |2t + 7| \geq 5\} = (-\infty, -6] \cup [-1, \infty)$.

23. Suppose you put \$5.21 into a jar on June 22. Then you added one penny to the jar every day until the jar contained \$5.95. Is the set $\{5.21, 5.22, 5.23, \dots, 5.95\}$ of all amounts of money (measured in dollars) that were in the jar during the summer an interval?

SOLUTION This set is not an interval because, for example, 5.21 and 5.22 are in this set but 5.215, which is between 5.21 and 5.22, is not in this set.

24. Is the set of all real numbers x such that $x^2 > 3$ an interval? Explain your answer.

SOLUTION The numbers 2 and -2 are both in the set $\{x : x^2 > 3\}$ because $2^2 = 4 > 3$ and $(-2)^2 = 4 > 3$. However, the number 0, which is between 2 and -2, is not in the set $\{x : x^2 > 3\}$ because $0^2 = 0 < 3$. Thus the set $\{x : x^2 > 3\}$ does not contain all numbers between any two numbers in the set, and hence $\{x : x^2 > 3\}$ is not an interval.